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UNIV. OF CALIFORNIA

THE ELEMENTS OF GEOMETRY, IN GENERAL TERMS,

WITH NOTES, &c. &c.

ALSO,

A VARIETY OF PROBLEMS AND THEOREMS,

CAREFULLY ARRANGED AND SELECTED;

WITH ANALYSIS.

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A NEW EDITION.

DUBLIN:
JOHN CUMMING, LOWER ORMOND QUAY.
MDCCCXLV.

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PREFACE.

THE general objects of the present treatise being to facilitate the study of the Elements of Geometry, as well as to promote the exercise of it amongst the students as much as possible; it seems advisable to show how far the reader may presume on these objects being attained. The former is attempted by giving the propositions of the Elements general demonstrations, which must, more than any other, contribute to give the reader clear and comprehensive views on the subject of Geometry, and to cultivate his reasoning powers with respect to all parts of human knowledge.

On the subject of *vivâ voce* demonstrations, much has been said against their general use, particularly in the higher orders of Mathematics and Physics, and indeed too much could not be said, if any good were likely to result from it. Academical reform is too hopeless a case, and one of too extensive a nature, to be worth discussing at any length.

In plane Geometry, however, where analytic artifice has no place, and where the reader is seldom, if ever, obliged to lose sight of the basis on which his reasoning rests, *vivâ voce* demonstrations are of powerful effect. They accustom him to clearness and solidity of reasoning, and enable him to take a view of the whole of a subject at

248128

once, the discussion of which would bewilder any person not accustomed to such.

The second object is aimed at, by presenting the reader with a collection of Geometrical *Theorems*, *Loci*, *Porisms*, and *Problems*, carefully selected from the best Geometrical writings, and some of them perhaps never before published. Mutual dependence has been considered as well as relative importance in the way of difficulty. The demonstrations are not given; first, from the great size and consequent expense it would entail upon the work; and secondly, because it would otherwise completely fail in the effect of exercising the student in reasoning for himself, and only accustom him to get off by rote the reasoning of others.

The best system of Geometry, that has lately been presented to the public, is that of Professor Leslie, *as far as he confines himself to the strict limits of plane Geometry*. As for the crude, inadequate and *false* demonstrations, that are huddled together at random in the books of Creswell and Bland, the reader is cautioned to avoid them, unless where he may be occasionally recommended to use them as being the works of some abler hands; for it may be said of these writers, as it has been said by an able critic of one of their cotemporaries, that "their errors only are original."

Perhaps the collection here presented is the most complete and comprehensive, that has yet appeared; and is by no means to be considered as a mere geometrical exercise: it is a collection of truths, and, as such, a powerful and necessary portion of human knowledge. They have a still greater use; "*humana notitia et humana potentia in idem coincidunt*," says Lord Bacon; they put the reader in possession of new resources, new instruments for the discovery of truth, and thus afford means of extending the portion of human knowledge, that they are connected with. It is by no means to be understood, that Geometry ought to be introduced into the higher branches of Mathematics; this would be a use of it totally out of its proper line, a use of it, that could only be attempted in modern times by a *Scots Professor*, anxious to lay hold on a claim to originality, one way or other, *in spite of his stars.* Let us recollect, that the Theorems of Geometry are translatable into *analytic language;* that, when so translated, they are capable of all the enlargement and extension, that analysis in general admits; and that in this way they may be said to exercise an indirect influence over the whole of Mathematics, of unbounded power and extensive use. Those remarks are not made at random; they must be acknowledged true by all acquainted with mathematical pursuits. How many Theorems, originally geometrical, have been generalized and carried to the highest perfection, and applied to the most important uses by analytic writers, that perhaps would otherwise have never been thought of! Need we instance the celebrated Theorems of Vieta, that have led to the *summation of series of sines and cosines of multiple arcs;* the *formation by induction of the expressions for the tangents of multiple arcs* by De Lagny; and Cotes's celebrated Theorem, which we may be certain was ultimately founded on geometrical reasoning, and is announced by the author as a Geometrical Theorem?

On such theorems as these, the whole of Analytic science is founded, and must continue to be founded, until analytic science is able to manage elementary truths; until it is able to bring under its dominion the principles of Trigonometry, if ever this æra shall arrive.* Every attempt then to decry Geometry, must be an attempt to put down all Mathematics, no less than that, which would attempt to reduce all to plane Geometry. Any person, that would make either attempt, is no mathematician, is a pretender, one that has fastened on some particular branch of Mathematics, rings endless changes upon the principles of it, and calls that Science. Such persons, however, can do but little mischief, as they in general have but small influence with others, and the best use that can be made of them, is to set them to worry one another. He only is the true Mathematician, who is ready to give each its proper share of weight, to leave them their full influence as a united and combined instrument of calculation, and not to attempt to sever that union, which affords them their chief strength.

Much more might be said on this subject; but it would be inconsistent with the plan of this work; so we shall forbear for the present, requesting the reader's indulgence for any oversights he may meet with.

Trinity College, *Sept*. 7, 1833.

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* In a book entitled, *Memoirs of the Analytic Society*, published at Cambridge in 1813, after proving the 47th of Euclid's first book by the theory of functions, the following remark occurs :—" It is only by this way of proceeding, or some analogous one, that we can ever hope to see the elementary principles of Trigonometry brought under the dominion of *Analysis*. It may suffice to have thrown out a hint, that may be followed up at some future opportunity."

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THE
FIRST BOOK
IN
GENERAL TERMS.

PROPOSITION 1, PROB.

On a given finite right line to construct an equilateral triangle.

With the extremities of the given line as centres and the given line as radius, describe two circles, (per post. 3); from a point, in which those circles intersect one another, draw lines to the extremities of the given one; those three lines form an equilateral triangle.

For each of the drawn lines is equal to the given line, as being radii of the same circle, and they are $\therefore =$ to each other. (per ax. 1.)

Note.—No more than two equilateral triangles can be formed on the same right line, viz. one on each side of it.

Those circles will intersect, because the centre of either being in the circumference of the other, part of its circumference must be within the other.

PROP. 2, PROB.

From a given point to draw a right line equal to a given finite right line.

Connect the given point with either extremity of the given line; on this connecting line form an equilat. trian.

(Prop. 1); with the connected extremity of the given line as centre, and the given line as radius, describe a circle, whose circumference shall meet the leg of the equilateral triangle, produced through the connected point; with the vertex of equilat. tri. as centre, and whole produced leg as radius, describe another circle to meet the other leg produced through the given point; this produced part is the line required to be drawn.

For it is = to the produced part of the other leg; since the whole produced legs are =, and the parts of them, which are the sides of the equilat. tri. are also = ; ∴ the remainders (viz. the produced parts) are = ; but the part produced from the connected point, is = to the given line, as being radii of the same circle ; ∴ the part produced from the given point, is = to the given line.

PROP. 3, PROB.

To cut from the greater of two given right lines a part equal to the less.

From either extremity of the greater draw a line = to the less (per Prop. 2,) and with the same extremity as a centre, and the drawn line as an interval, describe a circle; this circle cuts from the greater line a part = to the less.

For the part cut off between the circumference and centre is = to the drawn line, which was made = to the lesser of the given lines ; ∴ this part is = to the lesser of the given lines.

PROP. 4, THEOR.

If two triangles have two sides of the one respectively equal to two sides of the other, and the angle contained by those two sides of the one equal to the angle contained by the two sides equal to them of the other, their bases or third sides shall be equal, also the angles at the bases shall be equal, viz. those opposite the equal sides, and the whole triangles shall be equal.

If the two triangles be so applied to one another, that the vertex of one may coincide with that of the other, and a side of one coincide with a side equal to it of the other, and the other two sides lie towards the same part; then, since the angles at the vertices are =, and the vertices coinciding, the sides about those angles must coincide, and since they coincide and are =, their other extremities must coincide; and since those extremities coincide, which are at the bases, the bases themselves must coincide (per ax. 10,) and ∴ be =. And since the bases (and other sides) coincide, the angles at the bases must be =, and the sides and angles coinciding and being respectively =, the whole triangles must coincide and be =.

PROP. 5, THEOR.

In an isosceles triangle the angles at the base are equal to one another; and, if the equal sides be produced, the angles below the base shall be also equal.

Assume points in the legs below the base, equidistant from the vertex, and, from these assumed points, draw transverse lines to the opposite extremities of the base; then there are two triangles formed, having a side of isosceles triangle and cut off seg. in one, respectively = to a side of isosceles triangle and cut off seg. in the other; ∴ (by the 4th) the bases (viz. the trans. lines) are =, and also the angles at the bases (viz. those contained by sides of isos. tri. and trans. lines, and those contained by cut off seg. and trans. lines) are respectively =; and, in the triangles below the base, the angles opposite the base are = (per demon.) and the sides containing them are respectively =; ∴ (by 4th) the angles below the base are =, and also the angles contained by the base and transverse lines; which, being taken from the angles contained by the sides of isos. tri. and transverse lines, leave the angles at the base = (per ax. 3.)

Cor. Hence every equilateral triangle is also equiangular.—Vide Elrington.

PROP. 6, THEOR.

If two angles of a triangle be equal, the sides opposite them are also equal.

For if they be not =, let one of them be greater, and from it cut off a part (towards one of the = angles) = to the lesser, and draw a transverse line from the point of section to the opposite angle; then there are two triangles formed, having two sides in each respectively =, and the contained angles also = (by hypoth.) ∴ the triangles are =, a part to the whole, which is absurd.

Cor. Hence every equiangular triangle is equilateral; vide Elrington.

PROP. 7, THEOR.

On the same right line and on the same side of it, there cannot be two triangles formed whose conterminous sides are equal.

If it be possible that there can, 1st. let the vertex of each fall without the other, and draw a right connecting their vertices; then this connecting line will be the base of two isosceles triangles, and the angles at the base must be = in each; but if they be = in one of these triangles, in the other, one of them must be much greater than the other, and also = to it, which is absurd, (one of them being greater and the other less than one of the former pair.)

2dly. If the vertex of one triangle be supposed to fall within the other, connect their vertices and produce either pair of = legs. Then the angles below the base of one isosceles triangle are = (by the 5th); and of the angles above the base of the other isosceles triangle, one of them is much greater than the other; but they are also =, which is absurd.

3dly. If the vertex of one triangle be supposed to fall upon a side of the other, it is evident, that two of the conterminous sides are not equal.

PROP. 8, THEOR.

If two triangles have two sides of the one respectively equal to two sides of the other, and also have the base of the one equal to the base of the other, the angles contained by the equal sides are also respectively equal.

For, if the = bases be so applied to one another, that the sides, respectively =, may be conterminous; and that the triangles lie towards the same parts; the vertices must fall upon one another (per prop. 7), and the = sides must coincide (per ax. 10), ∴ the angles must coincide, and ∴ be =.

Sch. It is evident, that the remaining angles, opposite the = sides, are also =, and also the triangles themselves.

PROP. 9, PROB.

To bisect a given rectilineal angle.

On the sides containing the angle, assume = portions toward the vertex; connect their extremities; and on the side of the connecting line remote from the given angle, describe an equilateral triangle; the right line, joining its vertex with that of the given angle, bisects said angle.

For thus there are two triangles formed, having a common side (viz. line connecting vertices), and assumed portion in one, respectively = to said common side and assumed portion in the other, and also the bases or third sides = (viz. sides of equilat. trian.) ∴ the angles contained by the common side and assumed portions are =, (per prop. 8,) which are the parts of given angle. ∴ &c.

Cor. By this prop. an angle can be divided into 4, 8, 16, &c. = parts, by bisecting each part.

PROP. 10, PROB.

To bisect a given finite right line.

Describe on the given right line an equilateral triangle; bisect the angle opposite the given line; the line bisecting this angle, (if produced to it,) will bisect the given right line.

For thus there are two triangles formed, having the bisecting line and side of equilateral triangle in one, respectively = to said bisecting line and another side of equilat. trian. in the other, and the contained angles = (by construction.) ∴ the bases are = (by 4th,) ∴ the given right line is bisected.

PROP. 11, PROB.

From a given point in a given right line to draw a perpendicular to the given line.

From the given point (on each side of it,) assume = portions; upon their sum describe an equilateral triangle, and connect its vertex with the given point; the connecting line is at right angles to the given line.

For thus there are formed two triangles, having the three sides of one respectively = to those of the other, ∴, &c.

Schol. In the same manner a perpendicular can be erected at one extremity of a given line, by first producing the line.

PROP. 12, PROB.

To draw a perpendicular to a given indefinite right line from a given point without it.

Assume any point on the other side of the line; and with the given point as a centre, and the interval between the given point and the assumed point as radius, describe a circle, cutting the given line in two points; bisect the intercept, and connect the given point with the extremities of it, and also with the point of bisection; this line, drawn to the point of bisection, is perpendicular to the given line.

This can be proved as the former (per 8th).

PROP. 13, THEOR.

When a right line, standing upon another, makes angles with it, they are either two right angles, or are together equal to two right angles.

If the right line, standing on the other, be perpendicular to it, the angles are both right; if not, draw a perpendicular to the same point; (per 11.) it is evident, that the angles thus formed, are = to the given angles: therefore, &c.

Cor. 1. If several right lines stand on the same right line at the same point, they make angles, which taken together are equal to two right angles.

Cor. 2. Two right lines intersecting one another make angles, which taken together are = to four right angles.

Cor. 3. If several right lines intersect one another at the same point, all the angles taken together are = to four right angles.

PROP. 14, THEOR.

If two right lines, meeting another right line at the same point and at opposite sides, make angles with it, which are together equal to two right angles, those two right lines form one continued right line.

For, let it be supposed that one of them be produced beyond the vertex of the angles, so as that the produced part shall not coincide with the other: then two angles will be formed = to two right angles (by 13); but those will be either greater or less than the two given angles, which is absurd.

PROP. 15, THEOR.

If two right lines intersect one another, the verticle angles are equal.

For they have a common supplement to two right angles.

PROP. 16, THEOR.

If one side of a triangle be produced, the external angle is greater than either of the internal opposite angles.

For, connect with the opposite angle the middle point of the side conterminous with the produced part, and produce this connecting line till the produced part is = to it, and draw a right line from the extremity of the produced part to the external angle, this line shall form with the bisected side of the triangle an angle (subtended by the produced part of bisecting line) = to the angle of the given triangle subtended by the other part of bisecting line. For the triangles, to which these angles, thus subtended, belong, have an angle in each =, as being vertically opposite, and the sides containing them = by construction. But the external angle is greater than the former angle, and ∴ greater than the latter. This external angle is proved greater than the other internal remote angle, by producing the opposite side beyond the vertex of this external angle, and proving, as before, the new external angle, thus formed, greater than that internal angle: but this external angle is = to the given one (by the 15th.)

Cor. 1. If, from any point, two right lines be drawn to the same right line, one of them perpendicular to it, and the other not, the perpendicular falls at the side of the acute angle.

For, if it fell at the side of the obtuse angle, then an acute angle would be greater than a right, which is absurd.

Cor. 2. Two perpendiculars cannot be drawn from the same point to the same right line.

For then an external angle would be = to an internal and opposite, which is absurd.

PROP. 17, THEOR.

Any two angles of a triangle are together less than two right angles.

Produce any side; then the external angle is greater than either of the internal remote angles; but the external, with the adjacent internal, are = to two right angles; ∴ this internal, with either of the other internal angles, are less than two right angles. Thus any two angles of a triangle can be proved less than two right angles.

Cor. 1. If any angle of a triangle be obtuse or right, the other two are acute; and if two angles of a triangle be equal to one another, they are acute.

PROP. 18, THEOR.

In any triangle, if one side be greater than another, the angle opposite the greater side is greater than the angle opposite the less.

From the greater side cut off a part = to the less and conterminous with it, and connect the extremity of the part cut off with the opposite angle. This connecting line will cut off from the angle opposite the greater side a part = (by the 5th) to an angle which is greater (by the 16th) than the angle opposite the lesser side, as being external to it; ∴ the entire angle opposite the greater side is much greater than the angle opposite the less.

PROP. 19, THEOR.

If in any triangle one angle be greater than another, the side which is opposite the greater angle is greater than the side which is opposite the less.

For, if it be not greater, it must be either = to it, or less than it. It is not = to it; for then the angles, subtended by them, would be =; nor is it less than it; for then the angle opposite to it would be less than the other; ∴ it must be greater.

Cor. If from the same point two right lines be drawn to the same right line, of which one is perpendicular to it and the other not, the perpendicular is the least.

PROP. 20, THEOR.

Any two sides of a triangle are together greater than the third.

Bisect the angle contained by the two sides; the bisecting line shall cut the third side into segments, which are severally less than their conterminous sides, since they are opposite to lesser angles; for the angle, subtended by either side, is greater than the angle subtended by the conterminous segment, since it is greater than the other half of bisected angle, as being external to it.

Hence it is evident, that the difference between two sides of a triangle is less than the third side.

PROP. 21, THEOR.

Two right lines drawn to any point within a triangle from the extremities of any side, are together less than the sum of the other two sides of the triangle, but contain a greater angle.

Produce either of the lines drawn within the triangle to meet one of the sides, then the whole produced line is less than the uncut side and conterminous segment of cut side; to each add the remaining seg.; then this produced line and added seg. are less than the sides of original triangle: it also can be proved in the same way, that the two drawn lines are less than the produced line and added seg. ∴ *à fortiori*, &c. Secondly, the angle contained by those drawn lines is external to an angle, that is external to the angle contained by the two sides of original triangle, and ∴ much greater than it.

PROP. 22, PROB.

Given three right lines, of which any two together are greater than the third, to construct a triangle, whose sides shall be respectively equal to the three given lines.

From any point draw a right line = to one of them; and from the extremities of this, draw right lines respectively = to the other two; and with these extremities as centres, and the lines drawn from them as intervals, describe two circles; and, from one of their intersections, draw lines to the centres (or the extremities of the first drawn line), and the required triangle is formed.

For those lines drawn from intersection are respectively = to the two last assumed lines, as being radii of the same circles; ∴ the three sides of the triangle are respectively = to the three given lines.

Cor. In this manner we may construct a triangle equilateral to a given one.

NOTE.—It is evident, that the circles must intersect, from the limitation of the problem.

PROP. 23, PROB.

At a given point in a given right line to make an angle equal to a given angle.

In the legs of the given angle take any two points and connect them; then construct on the given line a triangle equilateral to the one thus formed, having the sides about the angle at the given point respectively = to those about the given angle.

Those sides will contain an angle = to the given angle, by the 8th.

PROP. 24, PROB.

If two triangles have two sides of the one respectively equal to two sides of the other, and if one of the angles contained by the equal sides be greater than the other, the side opposite the greater angle is greater than the side opposite the lesser angle.

From the vertex of the greater angle draw a line, making with the side which is not the greater* an angle = to the lesser; make this drawn line = to the side which is not the lesser; connect its extremity with the extremity of the side which is not the greater; this connecting line will be = to the side subtending the lesser angle (by 4th.) But it is less than the line subtending the greater angle; for if you join the extremities of it and the line subtending the greater angle, it shall be opposite a smaller angle; viz. to part of an angle at the base of an isosceles triangle, whilst the side subtending the greater angle is opposite to an angle greater than the other angle of the isosceles triangle at the same base.

* The reason why the angle is made with the side which is not the greater, is fully explained in the notes of Dr. Elrington's Euclid, page 150.

PROP. 25, THEOR.

If two triangles have two sides of the one respectively equal to two sides of the other, and if the third side of the one be greater than the third side of the other, the angle opposite the greater side is greater than the angle which is opposite the lesser.

For if it be not greater, it is either = to it or less than it. It is not =, for then the sides opposite to them would be =; nor is it less, for then the side which is given greater, would be less than the one given less. Therefore, since it is neither = nor less, it must be greater.

PROP. 26, THEOR.

If two triangles have two angles of the one respectively equal to two angles of the other, and a side of the one equal to a side of the other, whether it be adjacent or opposite to these equal angles, the remaining sides and angles are respectively equal.

First, let the = sides be adjacent to the = angles in each, the sides opposite to either of the = angles shall be =; for if not, cut off from the greater, a part = to the less, and conterminous with the given = side; and draw a right line from the extremity of that part to the vertex of the opposite angle; this drawn line will cut off from the angle to which it is drawn, an angle = to that angle of the other triangle, which is = to the entire of this cut angle, which is absurd. 2nd. Let the = sides be opposite to the = angles; it may in like manner be proved, that the sides between the = angles are =; for if they were not, an external angle of a triangle would be = to an internal remote.

Schol. It is evident that the triangles themselves are equal.

Cor. 1. In an isosceles triangle, the right line drawn from the vertex perpendicular to the base, bisects the base and vertical angle.

For the triangles thus formed have two angles, and one side in each respectively =, ∴ the other sides and angles are respectively =, viz. the segs. of base, and of vertical angle.

Cor. 2. It is evident that the right line which bisects the vertical angle of a triangle, bisects the base and is perpendicular to it, and that the right line drawn from the vertical angle bisecting the base is perpendicular to it, and bisects the vertical angle.

PROP. 27, THEOR.

If a line intersect two right lines, and make the alternate angles equal to each other, these two right lines are parallel.

For if not, let them meet on either side, then the external angle of the triangle is greater than the internal remote, but it is also = to it, which is absurd: ∴ the lines will not meet, if produced this way; and in like manner, it can be proved that they will not meet, if produced the other way; ∴ they are parallel.

PROP. 28, THEOR.

If a right line intersect two right lines, and make the external angle equal to the internal and opposite on the same side of the line, or make the internal angles on the same side together, equal to two right angles, the two right lines are parallel to one another.

For, in the first case, the alternate angles are =, as one of them is = to the external angle (by the 15th,) ∴ the lines are parallel. And in the second case, the alternate angles are =, since one of them with the adjacent internal is = to two right angles, and this adjacent internal with the other internal, also = to two right angles. ∴ &c.

PROP. 29, THEOR.

If a right line intersect two parallel right lines, it makes the alternate angles equal; and the external angle equal to the internal and opposite on the same side, and also the two internal angles at the same side together equal to two right angles.

For, if the alternate angles be not =, let one of them if possible be greater than the other; then if the lines be produced towards the side of the lesser angle, they will meet, (by ax 12,) since it, with the internal on the same side, would then be less than two right angles; but since the lines are parallel, they cannot meet, ∴ the alternate angles are not unequal. And since the alternate angles are =, the external, which is = to one of them (by the 15th), is = to the internal and opposite on the same side. Also, since the alternate angles are =, the two internal angles are together = to two right angles, for one of the alternate angles with the adjacent internal are = to two right angles, (by the 13th.) ∴ &c.

PROP. 30, THEOR.

If two right lines be parallel to the same right line, they are parallel to one another.

Let a right line be drawn cutting the three given lines, this drawn line makes with each pair of parallel lines an external angle = to an internal and remote on the same side of the line, and ∴ does so with the pair to be proved parallel.

PROP. 31, PROB.

Through a given point to draw a right line parallel to a given right line.

In the given line take any point; connect the given point with it, and through the given point draw a line making an angle with the connecting line = to an angle made by connecting line with given line, but on the contrary side of it. This second drawn line is parallel to the given line, since the alternate angles are equal.

PROP. 32, THEOR.

If any side of a triangle be produced, the external angle is equal to the sum of the two internal and opposite angles; and the three internal angles of any triangle taken together, are equal to two right angles.

Draw through the external angle, a line parallel to the side, not conterminous with the produced part, (i. e. to the side opposite the adjacent internal angle ;) it divides the external angle into two parts respectively = to the internal and remote angles, (by the 29th,) therefore, &c. And since the external angle is = to the two internal and remote angles and the external with the adjacent internal = to two right angles, the adjacent internal, with the two internal remote angles are together = to two right angles.

Cor. 1 Hence in a triangle, if one angle be right, the other two are together = to a right angle ; and if one angle be = to the other two, it is a right angle.

Cor. 2. If two triangles have two angles = to two, the remaining angles are also equal.

Cor. 3. In a right angled isosceles triangle, each angle at the base is half a right angle.

Cor. 4. Each angle of an equilateral triangle is = to a third part of two right angles.

Cor. 5. Hence can be derived a method of trisecting a right angle; by assuming any portion on one of the legs from the vertex and constructing on it an equilateral triangle, and bisecting the angle of it cut off from the given right angle.

Cor. 6. All the internal angles of any rectilineal figure, together with four right angles, are = to twice as many right angles as the figure has sides.

For if you assume any point within the given figure and draw lines from each of the angles to it, the figure will be divided into as many triangles as it has sides ; and therefore all their angles taken together are = to twice as many right angles as the figure has sides, (prop. 32,) but still the angles at the assumed point are = to four right angles (cor. 3. prop. 13.) ∴ the angles of the figure together with four right angles are = &c. &c.

This also can be proved by drawing lines from any angle of the figure to all the others except the two adjacent ones; then the figure will be divided into as many triangles as it has sides — 2; and the angles of each of the triangles into which it is divided are = to 2 right angles, ∴ &c.

Cor. 7. The external angles of any rectilineal figure are together = to four right angles. For each external angle, with the adjacent internal, is = to two right angles, (prop. 13.) Therefore, all the external angles, with all the internal, are = to twice as many right angles as the figure has sides; but the internal angles, with four right angles, are = to twice as many right angles as the figure has sides; take away from both the internal angles, and the external angles remain = to four right angles.

PROP. 33, THEOR.

Right lines which join the adjacent extremities of equal and parallel right lines are also themselves equal and parallel.

Draw a diagonal, then two triangles are formed, having a given = side, and diagonal in one respectively = to the other given = side, and same diagonal in the other, and the contained angles = as being alternate, ∴ the third sides, or joining lines are =, and also the angles at them; and since the angles contained by them and diagonal are =, the joining lines are parallel.

PROP. 34, THEOR.

The opposite sides of any parallelogram are equal to one another, as are also the opposite angles, and the parallelogram itself is bisected by the diagonal.

For in the two triangles formed by the diagonal; the alternate angles are respectively = to one another, (prop. 29,) and the side adjacent to the = angles, viz. the diag. is common to both triangles, ∴ the other sides are respectively = (prop. 26,) viz. the opposite sides; and the triangles themselves are =, and the opposite angles, which are subtended by the diagonal, are, =; also the other pair of opposite angles must be = since either of them is a supplement to either of the former.

Cor. 1. If one angle of a parallelogram be a right angle, the other three are right angles. For the adjacent angle is right, because with a right angle it is = to two right angles, and the opposite are right angles, because they are = to these right angles.

Cor. 2. If two parallelograms have one angle = in each, the other angles are respectively =. For the angles opposite to these = angles are = to them, and therefore to each other (ax. 2.) but the angles adjacent to them are also = to each other, as being together with these equal angles = to two right angles (prop. 29.)

PROP. 35, THEOR.

Parallelograms on the same base and between the same parallels are equal.

Connect the sides opposite the base, (if necessary,) then there are two triangles formed (viz. the excesses of the quadrilateral figure above each parallelogram,) having two sides of one respectively = to two sides of the other, and the contained angles = since they are the differences of = angles (two of which are formed by producing the common base) ∴ the triangles are =; if one of those triangles be taken from the whole figure, it will leave one par., and the other being taken away, leaves the other par., which parallelograms must ∴ be =.

PROP. 36, THEOR.

Parallelograms upon equal bases and between the same parallels, are equal.

For, if the extremities of the base of one of them be connected by right lines with the adjacent extremities of the side opposite the base of the other, those connecting lines, with the connected lines, form a parallelogram = to either of them, ∴ they are = to one another.

PROP. 37, THEOR.

Triangles on the same base, and between the same parallels, are equal.

Draw through one extremity of the base (to meet the line parallel to the base,) lines respectively par. to the sides of the triangles terminating in the other extremity of the base; thus = parallelograms are formed (35. 1.) of which the given triangles are halves (34. 1.) and are ∴ equal.

PROP. 38, THEOR.

Triangles on equal bases and between the same parallels, are equal.

Draw through either extremity of each base (to meet the line par. to the bases) lines par. to the sides of the triangles terminating in the other extremity of each base: thus there are two = parallelograms formed, of which the given triangles are halves; they are ∴ equal.

PROP. 39, THEOR.

Equal triangles on the same base and on the same side of it, are between the same parallels.

For, if they are not, draw through the vertex of one of them a line par. to the base, it cuts a side of the other or it produced; connect the point of section with the opposite angle; the triangle so formed shall be = to a triangle, (37. 1.) which is = to a triangle, that must be either greater or less than this, which is absurd; ∴ they must be between the same par.

PROP. 40, THEOR.

Equal triangles on equal bases and on the same side, are between the same parallels.

This proposition may be demonstrated precisely as the former (from the 38th.)

PROP. 41, THEOR.

If a parallelogram and a triangle have the same base, and are between the same parallels, the parallelogram is double of the triangle.

For, by drawing a diagonal, you divide the parallelogram into two = triangles, either of which is = to the given triangle; ∴ the parallelogram is double of the triangle.

Schol. Hence it is evident that a parallelogram is double of a triangle, if they have = bases and are between the same parallels.

Cor. If a parallelogram and a triangle be between the same parallels, and the base of the triangle be double that of the parallelogram, the parallelogram is = to the triangle.

For, if the base of the triangle be bisected, and the point of bisection be connected with the vertex, it is divided into = triangles, either of which is = to half the parallelogram; ∴ the parallelogram is = to the whole triangle.

PROP. 42, PROB.

To construct a parallelogram, equal to a given triangle, and having an angle equal to a given one.

Through the vertex of the triangle draw a right line par. to the base; bisect the base; and at the point of section, with either segment, draw a line (to meet the line parallel to the base), making with the segment an angle = to the given one; and through the other extremity of this seg. draw a line par. to the one making the required angle; this parallelogram, thus formed, is = to the given triangle; for it is double of half the triangle (41. 1.), and it has an angle = to the given one.

PROP. 43, THEOR.

In a parallelogram, the complements of the parallelograms about the diagonal, are equal.

For, the whole parallelogram is divided, by the diagonal, in = triangles; ∴ if, from each of the large triangles, ou take away the small ones, which respectively belong o them, the complements will remain =.

By the complements are meant those parallelograms hich, with the ones about the diagonal, make up the hole parallelogram.

Cor. The parallelograms about the diagonal of a square, re squares.

For the whole square being divided by the diag. into ight angled isosceles triangles, and the parallelograms, bout the diagonals being equi-angular to those, ∴ they ave one angle in each right, and each of the others half f a right angle; ∴ the adjacent sides are = and also he opposite sides; ∴ the figures are equilat. to each ther, and also rectangular.

PROP. 44. THEOR.

To a given right line, to apply a parallelogram equal to a given triangle, and having an angle equal to a given one.

If the base of the given triangle and the given right line be not *in directum;* produce the given line until the produced part becomes = to the base of the triangle, and n this produced part, construct a triangle = to the given ne; (cor. prop. 22.) on the half of this base adjacent to he given line, construct a parallelogram = to the given riangle and in the given angle; then complete the parallelogram of which the given line and side of first parallelogram are sides, draw the diagonal of it through the ngle contained by those sides, and produce it to meet he opposite side of the first formed par^{m}. produced; hrough the point of occurse draw a line par. to the given ine, to meet the opposite side of second par^{m}. produced, nd produce to meet this par. line, the side of first par^{m}. djacent to the given line; then there is formed on the iven line a par^{m}. which with the first formed par^{m}. is omplemental of the two about the diagonal of the entire ar^{m}. and is ∴ = to first par^{m}. (43, 1,) and it has an ngle = to the given one, since it is external to an angle, that is internal to one = to the given one; the quality of those angles is evident from prop. 29, 1.

PROP. 45. THEOR.

To construct a parallelogram equal to a given rectilinear figure, and having an angle equal to a given one.

Resolve the given figure into triangles: construct a parallelogram = to one of them, and having an angle = to the given one; on the side of this par^{m}. not about the angle = to given one, construct a par^{m}. = to another of the triangles, and in the given angle, external to the one = to the given one in the first; and so on until par^{ms}. are formed = to all the triangles into which the figure is resolved. The figure made up of all these, is a parallelogram = to the given figure, and in the given angle. For, the side of the first, about the given angle, and conterminous with that side of it, to which the second is applied, is *in directum* with the conterminous side of the second, since they make, with the same line, at the same point, angles which are equal to two right ones; hence, obviously, the fig. is a parallelogram, and, by construction, equal to the given right lined figure, and having an angle = to the given one.

PROP. 46. PROB.

On a given right line, to construct a square.

To either extremity of the given line, draw a perpendicular, and make it = to it, (prop. 11, sch. and prop. 3,) through the extremity of this per., remote from given line, draw a right line, parallel to given line; and through the other extremity of given line, draw a right line, parallel to perpen., to meet the first par. line: those lines form a □^{2}.

Because the figure thus formed is a parallelogram, and one of its angles being right, the remaining angles must be so (cor. 1, prop. 34); and because the perpendicular and given line, which are adjacent, are =, their opposite sides are also = to them (prop. 34); ∴ the figure is equilateral; and it has been proved rectangular; it is ∴ a square.

Cor. 1. The squares of equal right lines are equal.

For, if a diagonal be drawn in each, a triangle, which is half of one □^{2}, can be proved = to a triangle which is half of the other □^{2} (prop. 4.); ∴ the squares themselves, which are the doubles of those triangles, are =.

Cor. 2. If two squares be equal, their sides are equal.

For, if not, cut off from conterminous sides of that which is supposed to have the greater sides, parts = to the supposed lesser sides, and connect their extremities; the triangle formed by those cut off parts, and con. line, is = to a triangle which is half the other $\square^2$, and ∴ to a triangle of which itself is a part, which is absurd.

PROP. 47. THEOR.

In a right angled triangle, the squares of the side subtending the right angle is equal to the sum of the squares of the sides which contain the right angle.

Describe squares on the sides of the triangle; and from the right angle draw a right line, dividing the $\square^2$ on the hypotenuse into two parallelograms. It can be proved, that each of those parallelograms is respectively = to the $\square^2$, on that side of the triangle which is adjacent to it. For, if the acute angle contained by that side, and the hypotenuse, be added to the angles of their $\square^2$rs which are adjacent to it, two obtuse angles will be formed, = to one another, the legs about which are also respectively = ; if, then, those legs be joined, the triangles thus formed shall be = ; but one of these triangles is half of one of the parallelograms into which the $\square^2$ on the hypotenuse is divided, and the other is half of the $\square^2$ on the side adjacent to this par. (prop. 41. 1.); ∴ this entire par^{m}. is = to the entire $\square^2$ on that side: in like manner, the other par^{m}. can be proved = to $\square^2$ of side adjacent to it; ∴ the whole $\square^2$ of hypotenuse, &c.

Cor. 1. Given in numbers, any two sides of a right angled triangle, the third side can be found; for it is the square root of the sum or difference of the squares of the given lines, according as the given sides contain the right angle or not.

Cor. 2. To find a square equal to the sum of two or more squares. Assume lines respectively = to the sides of any two $\square^2$rs, with these form a right angle, and connect their extremities; the $\square^2$ of the joining line is = to the sum of the $\square^2$rs of the assumed lines. Then, with this joining line, and a side of another $\square^2$, form a right angle; the $\square^2$ of the line joining their extremities is = to the sum of the three $\square^2$rs, and so on.

Cor. 3. To find a line, whose square is equal to the difference between two squares.

Produce the greater of the lines, till the produced part becomes = to the lesser; from their common extremity as centre, and the greater as radius, describe a circle, and from the other extremity of produced part, erect a perpendicular to meet its circumference; the $\square^2$ of this perpendicular is = to the required difference, because the $\square^2$ of it, with the $\square^2$ of the lesser, is = to the $\square^2$ of the greater; $\therefore$, &c.

Cor. 4. If from any angle of a triangle a perpendicular be let fall on the opposite side, the difference between the squares of the sides which contain that angle, is equal to the difference of the squares of the segments of the sides on which the perpendicular falls.

For, the $\square^2$ of one side is = to $\square^2$ of adjacent seg. and $\square^2$ of per.; and the $\square^2$ of the other side is = to $\square^2$ of adjacent seg. and $\square^2$ of per. If the $\square^2$ of perpendicular be taken away, it is evident that the dif. between the $\square^2$rs of the sides must be = to the dif. between the $\square^2$rs of the segments.

Cor. 5. The excess of the square of either side, above the square of the conterminous seg. of third side, is the same. For, it is the $\square^2$ of the perpendicular.

PROP. 48. THEOR.

If the square of one side of a triangle be equal to the sum of the squares of the other two sides, the angle opposite to that side is a right angle.

From the vertex of this angle, raise a perpendicular to either of the sides containing it, and = to the other side; connect the extremity of this perpendicular with the extremity of the side to which it is a perpendicular: the square of this connecting line is = to the square of the third side of the given triangle. Therefore these two triangles are equilateral to one another, and therefore equiangular. Therefore, the angle required to be proved so, is a right angle.

OBSERVATIONS.

1st. In the fifth, the equality of the angles below the base seems to be introduced merely to prove that of the angles at the base; but the equality of the angles at the base can be proved without producing the sides.

Fig. 1. Plate 1.

On the = sides AB, AC, assume = portions, AD, AE; join DC, EB, and DE. Then in the triangles BAE, CAD; the sides BA and AE are respectively = to CA and AD, and the angle A common; ∴ DC and BE are =, and the angle ABE is = to ACD. Then in the triangles DBE, ECD, the sides DB, BE, are respectively = to EC and CD, and the angle DBE to ECD; ∴ the angles CED, BDE, are =, and also the angles BED and CDE; ∴ the angles BDC and CEB are =. Also, in the triangles BDC, CEB, the sides BD and DC are respectively = to CE and EB, and the contained angles =; ∴ the angles DBC and ECB are =.

2nd. In the sixth (which is the converse of the fifth), we must cut off the = part, towards the base; for if we cut it off towards the vertex, we would not have the necessary sides or angles.

3d. In the seventh, we mean that if one pair of conterminous sides be =, the other pair cannot be so. This proposition is made use of only in proving the eighth. It does not follow from this prop. that there cannot be on the same base, two triangles, whose legs are respectively =; but the = legs must not be conterminous.

4th. The eighth is the second case of = triangles, and is the converse of the fourth.

5th. In the ninth, the equilat. tri. is constructed on the side of the connecting line, remote from the given angle, least its vertex should coincide with the given angle.

6th. In the eleventh, if the point be given in the extremity of the line, you may first produce the line on that side.—Vide Deducibles, prop. 12.

7th. In the twelfth, we say an indefinite line, for otherwise the point might so lie, that we could have no perpendicular on it from the given point.

8th. In the fourteenth, we say, at different sides; for if they met at the same, they could not lie in directum.

For, it is evident that two right lines might meet another at the same point, and make angles with it, together = to two right angles (for one of them might be as obtuse, as the other is acute,) and yet not be *in directum*, if they both be at the same side of the line which they meet.

9th. In the fifteenth, by supplement is meant, what either of the vertically opposite angles wants of two right angles.

10th. In the twentieth, the difference between two sides is less than the third; for if it was =, then one side would be = to the other two; and if greater, one side would be greater than the other two.

11th. In the twenty-fourth, we say, with the side which is not the greater, not with the lesser, because they might be =. We take this side to avoid the different cases, which taking the other might involve; as, if the other were taken, then the extremity of the new drawn line might fall above, below, or upon the base; but, by taking the side not the greater, it must fall below it, as is proved in the notes to Dr. Elrington's Euclid.

12th. In Cor. 7, of thirty-second, we must not produce the sides consecutively; i. e. no two of them are to be produced from the common extremity.

13th. Conterminous, means having a common point of termination.

14th. Another way for shewing the equality of triangles is, when they have two sides, respectively =, and the contained angles supplemental.

15th. When two triangles have two sides, mutually =, but the contained angles either greater or less than two right angles, that triangle will have the greater area, whose angle, contained by those sides, approaches nearer to a right angle. This is evident from Prop. 15 of Deducibles.

16th. The distinction between coincident and = triangles is: coincident triangles are those whose sides and angles are mutually =; and = triangles, those whose areas only are =.

17th. If we bisect the diagonal of a parallelogram, and draw a line through the point of bisection, this line will bisect the parallelogram.

18th. The difference between a problem and a theorem is, that a problem consists of the data and quæsita, and requires solution: a theorem consists of the hypothesis, and the thing asserted, and requires demonstration.

19th. In the second, the first leg produced is the leg of the triangle which terminates at the first centre; the second is that which terminates at the given point.

If the point be so situated, that the lines drawn from it to the extremity of the base, form an equilateral triangle, the thing is done; for, then either of them is = to the given line.

If the point is in the line, then with the part of the line, intercepted between the point and either extremity of the given line, form an equilat. tri., and proceed as before.

If the point occurs in the extremity of the given line, it is not necessary to construct an equilat. tri.; but from the given point as centre, and given line as radius, describe a circle: any line, drawn from the given point to meet the periphery of this, will be = to the given line.

TABLE OF SIGNS.

=	Sign of Equality.
+	Sign of Plus, or Addition.
×	Sign of Rectangle, or Multiplication.
-	Sign of Minus, or Subtraction.
$\square^2$	Sign of Square.
∴	Sign of Therefore.

A VARIETY

OF

THE MOST IMPORTANT

PROBLEMS AND THEOREMS,

DEDUCIBLE

FROM THE FIRST BOOK,

DONE ANALYTICALLY.

PROP. 1. PROB.

To draw from a point (A) *given in the side of a triangle* (DEF) *a right line bisecting said triangle.*

Fig. 2.

ANALYSIS.

Suppose it to be done, and that the right line AB bisects the given triangle. Bisect the side DF in C (10. 1. of Elr.); join AE, CE and CB. Then the triangles FEC and CED are = (38. 1. of Elr.); ∴ FEC is half DEF, and is consequently = to FBA; if FBC be taken away, the remaining triangles BAC and CBE are =, and they, being = and standing on the same base, must be between the same parallels (39. 1. Elr.). Therefore, it is evident, that, if you bisect DF in C, join AE and draw CB parallel to it, the straight line drawn from the given point A to B, bisects the triangle.

For, join CE; then the triangles CEB and ABC are = (37. 1 Elr.); ∴ if you add the common part FCB to both, the whole BAF is = to the whole EFC; ∴ &c.

PROP. 2. PROB.

To draw through a given point (D) *a right line making equal angles with two lines* (AB *and* AC) *given in position.*

Fig. 3. ANALYSIS.

Suppose it done, and that GF is the required line. Draw through the external angle EAC, AK parallel to GF (31. 1. of Elr.). Then the alternate angles KAI, AIF are = (29. 1. Elr.) and also the external angle EAK is = to the internal AFI; ∴ the angles EAK and KAC are =, and the line KA bisects the whole angle EAC.

Therefore it is evident, if you bisect the angle EAC by AK and draw through D, GF parallel to AK, that the angles AFI and AIF, are =.

For AIF and IAK are = and EAK and AFI are also = (29. 1. Elr.) ∴ AIF and AFI are =.

PROP. 3. PROB.

From a point (C) *given in the side of a parallelogram* ADEB *to draw a line bisecting it.*

Fig. 4. ANALYSIS.

Suppose it to be done, and that the line CF bisects it; draw the diagonal BD. Then the triangle DBE is half the parallelogram (by 34. 1. of Elr.); also the figure FCBE is half of it; ∴ if you take away the common figure FGBE, the triangles DGF and BGC are =; it is also evident that they are equiangular; ∴ they are equilateral; ∴ the diagonal is bisected by the line CF. Then if you bisect the diagonal, and draw from the given point a line through the point of bisection, it bisects the parallelogram.

PROP. 4. PROB.

To draw to the same point, in a right line AB *given in position, two equal right lines from two given points* (D and E) *without it.*

Fig. 5. ANALYSIS.

Suppose it to be done, and that DF and EF are the lines required to be drawn; connect the given points; it is evident, that the angles FDE and FED are =; bisect the angle DFE by FC; then it is evident, that the angles

FCE and FCD are =, and are ∴ right angles; and also that DC is = to CE (by 4. 1. Elr.).

Then it is evident, that, if the points are connected, the connecting line bisected and a perpendicular be drawn from the point of bisection to meet the given line, the point, in which it meets it, is the point to which the = lines can be drawn.

NOTE. The points must be so situated that the line connecting them cannot be at right angles to given line.

PROP. 5. THEOR.

Equal triangles (ABC and DEF) *between the same parallels stand on equal bases.*

Fig. 6.

For, if they do not, cut from the greater BC a part GC = to EF; connect GA.

Then the triangles GAC and EDF are = (38. 1. Elr.); ∴ GAC is = to ABC, a part to the whole, which is absurd.

PROP. 6. THEOR.

If one side (AB) *of any triangle be bisected, and a right line* (DF) *be drawn through the point of bisection* (D) *parallel to the base, it bisects the third side.*

Fig. 7.

ANALYSIS.

Suppose it to be the case; connect FB; then since AF is = to FC, the triangles ABF and FBC are =; ∴ if FBC be proved = to half of ABC, the theorem is proved.

Join DC; since AD is = to DB, the triangles ADC and BDC are =; ∴ BDC is half ABC; but BDC and CFB are = since they are on the same base and between the same par.; ∴ BFC is half ABC, and ∴ (by the foregoing paragr.) AC is bisected by DF.

PROP. 7. THEOR.

If a right line (DF) *bisects any two sides of a triangle, it is parallel to the third side. And the triangle* (ADF) *cut off by it, is the fourth part of the whole triangle.*

Fig. 7.

1st. If DF be not par. to BC, draw DG parallel to it; then, since AB is bisected in D, and DG parallel to the base, AC is bisected in G; but it is also bisected in F, which is absurd; thus it can be proved that no line but DF can be drawn through the points D or F parallel to BC.

2d. Since AB is bisected in D, the triangle ADF is = to BDF, and is ∴ half of ABF; but ABF is half of ABC; ∴ ADF is the fourth part of ABC.

PROP. 8. THEOR.

In a right angled triangle (ABC) *the right line joining the point of bisection of hypotenuse, with the right angle* (CBA) *is equal to half the hypotenuse.*

Fig. 8.

ANALYSIS.

Suppose it to be the case. Then since AD is = to DB, the angles DBA and DAB are =. Draw DE parallel to BC; then (by prop. 6. of analy.) AE and EB are =, and the angles AED and DEB are also =; ∴ AED is a right angle; ∴ the triangles AED and DEB are equal in every way; ∴ AD, DB and DC are = to one another.

Cor. Thus it is evident, that the angle in a semicircle is a right angle. Since from the centre D a circle might be described (through the three points C, B, A,) of which AC would be a diameter.

Or, it may thus be proved that the angle in a semicircle is a right angle.

Since DB and DC are =, the angles DCB and DBC are = (5. 1. Elr.); for the same reason the angles DBA and DAB are =; ∴ the angle CBA is = to the sum of BCA and BAC; ∴ (by cor. 1. 32. 1. Elr.) CBA is a right angle.

PROP. 9. PROB.

To draw from two given points (G *and* H) *right lines making equal angles at the same point of a line* (AB) *given in position.*

Fig. 9.

ANALYSIS.

Suppose it to be done, and that GD and HD are the lines, draw GI at right angles to AB, and produce it to meet HD (produced) in C; then because the angle GDA is = to HDB, it is also = to IDC, which is vertically opposite to HDB, and GID is = to CID being both right angles, and ID is common to the two triangles DCI and DGI; ∴ GI is = to IC.

Therefore, it is evident, if you let fall a perpendicular GI (from either of the points) on the given line and produce it, until the produced part IC is = to GI, and connect the extremity C with the other given point H; the point D, where the connecting line cuts the given line, is the point required. For, join GD; since GI is = to IC and ID common and the angles GID and CID =; the angles IDG and IDC are = (prop. 4. 1. Elr.); but IDC is = to HDB (prop. 15. 1. Elr.); ∴ HDB and GDI are =.

PROP. 10. THEOR.

If the right lines (GD and HD) *drawn from two given points, make equal angles with a right line* (AB) *given in position, they are the shortest lines, that can be drawn from those points to any one point in that line.*

Fig. 9.

For, if possible, let GE and HE be drawn less than them; draw GI at right angles to AB and produce till it meets HD produced in C; it is evident, that DC is = to DG (26. 1. Elr.) and ∴ that the whole HC is = to GD and DH; it is also evident, that, if EC be drawn, it is = to GE, and ∴ that GE and EH are = to HE and EC; but HC is less than HE and EC; ∴ GD and DH are less than GE and EH; in like manner, they can be proved less than any other two lines, &c.

PROP. 11. PROB.

To find a point in the side (AC) *of a triangle,* (ABC) *from which a line may be drawn to another side* (AB) *parallel to the third side, and equal to that segment* EB *(of the side to which it is drawn) that is conterminous with the parallel side* (BC.)

Fig. 10.

ANALYSIS.

Because DE is par. to BC, the angles EDB and DBC are =; and because DE is = to EB, the angles DBE and BDE are =; therefore the angles DBC and DBE are =.

Therefore it is evident, if the angle ABC, opposite the line in which the point is to be found, is bisected by BD, that D is the required point. For, since DE is par. to BC, angle DBC is = to BDE; ∴ BDE is = to EBD, and ∴ DE is = to EB.

PROP. 12. PROB.

To draw a perpendicular to the extremity (C) *of a given line* (AC) *without producing it.*

Fig. 11.

Take any point B in the given line, and from it draw BD at right angles and = to CB, and through D draw DE at right angles to BD; draw BE, bisecting the angle CBD, to meet DE; join EC; EC is at right angles to AC.

For in the triangles EBD and ECB, the sides CB and BE are respectively = to DB and BE, and the angle DBE is = to CBE; ∴ the angle BDE is = to the angle BCE (4. 1. Elr.); but BDE is a right angle; ∴ BCE is a right angle.

NOTE.—DE and BE will meet, since the angles DBE and BDE are less than two right angles, (ax. 12).

PROP. 13. PROB.

To draw through a given point, (A) *placed between two given right lines,* (CD, CB) *that are not parallel, a right line terminated by the given lines, and bisected in the given point.*

Fig. 12.

ANALYSIS.

Suppose it done; then through A draw AF par. to CB, and AG parallel to CD; because DA is = to AB, and the angles DAF and ABG =; as also the angles DFA and AGB; (prop. 28. 1. Elr.) the triangles DFA and AGB are = in every respect; ∴ FA is = to GB.

Therefore it is evident, that, if through the given point, lines AF and AG are drawn parallel to the given ones, CD and CB; and if GB be taken = to CG or FA, that the right line BD, drawn from B through the given point, is terminated by the given lines and bisected in the given point (prop. 26. 1. Elr.).

PROP. 14. THEOR.

If a right line, (CD) *drawn from the vertex of any triangle,* (CAB) *bisect the base, it bisects every line parallel to the base.*

Fig. 13.

For, if it do not, draw CH to bisect EF (par. to the base.) Join ED, FD and HD. Then, because EH and HF are =, the triangles CHE and CHF are = (prop. 38, 1. Elr.); also the triangles EHD and FHD are = ; ∴ the figures CHDF and CHDE are =; but the triangles CDB and CDA are = ; and also the triangles DFB and EDA ; ∴ CDF and CDE are = ; then CDF is half CEDF, and also CHDF is half CEDF ; ∴ CDF and CHDF are =, which is absurd ; ∴ EF is not bisected in H ; and, in like manner, it can be proved, that it cannot be bisected in any point but G.

PROP. 15. THEOR.

Of all the triangles, that can be formed having two sides of the one respectively equal to two sides of the other, the greatest is that, which has those two sides at right angles to one another.

Fig. 14.

ANALYSIS.

Since the triangle DAB, which has the side DB (= to CB) at right angles to the common side AB, is greater than the triangle CAB, the line CD, joining their vertices, will not be parallel to AB ; for, if it were, the angle CDB would be right, since ABD is right ; then CB would be greater than DB ; but it is = to it ; ∴ the line CE, drawn from C par. to AB, must fall below the point D ; then it is evident, if you join EA, that the triangle EAB, which is = to CAB, (prop. 37, 1. Elr.) is less than DAB.

PROP. 16. THEOR.

Of all the triangles, having the same vertical angle, and whose bases pass through a given point (A), *the least is that* (viz. CBD), *whose base* (BD) *is bisected in the given point.*

Fig. 12.

Let CEI be any other triangle, whose base EI passes through the given point, and is not bisected in it ; it is greater than CBD.

Draw BH parallel to CD ; then, since BA and AD are = (by hyp.) and the angles BAH and IAD are =, (prop. 15. 1. Elr.) and the angles AID and AHB are =, (prop. 29. 1. Elr.) the triangles BAH and IAD are = to one another ; to each add the figure BAIC ; then BDC and BHIC are = ; ∴ CEI is greater than CBD.

PROP. 17. THEOR.

If two finite right lines (DE and BC) *bisect one another, the right lines,* (DB, BE, EC, CD) *joining their extremities, form a parallelogram.*

Fig. 15. ANALYSIS.

Since DBEC is a parallelogram, the triangles DAB and CAE are =, and the angles AEC and ADB are also =, and also the joining lines are respectively =.

Then since, in the triangles DAB and CAE, the sides DA and AB are respectively = to CA and AE, and the angles DAB and CAE, are =, (prop. 15. 1. Elr.) the triangles themselves are =, and the side DB is = to CE, and the angle BDE is = to DEC; ∴ DB is parallel to CE: in like manner it can be proved, that DC is par. and = to BE; ∴ DBEC is a parallelogram.

Cor. It is also evident, that, if two equal right lines bisect one another at right angles, the right lines joining their extremities, form a square.

PROP. 18. THEOR.

If, from one side (AB) *of a scalene triangle,* (ABC) *a part* (AD) *be cut off, equal to another side,* (AC) *and the point of section be connected with the opposite angle; the angle contained by this connecting line and third side, is equal to half the difference of the angles at the third side.*

Fig. 16.

Since AD is = to AC, the angles ADC and ACD are =; but the angle ADC is = to DBC + BCD; ∴ ACD is = to DBC + BCD; ∴ the whole angle ACB is = to ABC + twice BCD.

PROP. 19. THEOR.

If a right line (DC) *be drawn, bisecting any angle* (ADB) *of a scalene triangle, so as to cut the opposite side* (AB), *the segment* (AC) *of the cut side, next the lesser angle, is greater than the segment next the greater angle.*

Fig. 17. ANALYSIS.

Since the angle DBA is greater than DAB, the side

DA is greater than DB: make DE = to DB, and join EC; the triangles DEC and DBC are = in every respect. But since AC is greater than CB, and ∴ than EC, the angle AEC is greater than EAC; this then is evidently the thing to be proved, in order to prove the proposition. Therefore, if you make DE = DB, and join EC, the triangles DEC and DBC are = in every respect; but the angle AEC is evidently greater than ECD, and ∴ than DCB; but DCB is greater than DAC; ∴ CEA is much greater than CAE; for which reason, the line AC is greater than EC, and ∴ than CB.

PROP. 20. PROB.

To find a point within a given triangle, (CHF) *from which right lines, drawn to the several angles, will divide the triangle into three equal parts.*

Fig. 18.

ANALYSIS.

Suppose I to be the point, and that the lines IC, IH, and IF trisect the triangle.

Through I draw IA and ID parallel to HC and HF; join HA and HD. Since AI is parallel to CH, the triangle CIH is = to CAH; but CIH is one-third of the entire; ∴ CAH is one-third; for the same reason FDH is one-third; ∴ ADH is one-third; ∴ the three triangles CAH, ADH and DFH are =, and of = altitudes; ∴ the bases CA, AD, and DF are =.

Therefore, if the base CF be trisected, and the lines AI and DI be drawn respectively par. to CH and FH, it is evident, that the point, in which they meet, is the required point; for the equality of the triangles on the same base and between the same parallels can be proved as in analysis; ∴ &c. &c.

PROP. 21. THEOR.

If right lines (AE *and* CE) *be drawn from the extremities of any side* (AC) *of a triangle to any point within it; they form, with the other two sides, angles, the sum of which is equal to the difference between the vertical angle* (ABC) *of the triangle and the angle* AEC *contained by the drawn lines.*

Fig. 19.

Draw BE and produce it to D: the angle AED is = to the angles EBA and EAB; ∴ EAB is the difference between AED and ABC; in like manner, ECB is the difference between DEC and DBC; ∴ &c.

PROP. 22. PROB.

Given an isosceles triangle (ABC) *to construct another on the same base with it* (AC) *having double its vertical angle.*

Fig. 19. ANALYSIS.

Suppose the angle AEC double the angle ABC, join BE and produce it to D. Then, because AEC is isosceles, the sides AE and EC are =, and also the sides AB and BC are =, and BE common to the triangles AEB and CEB; ∴ (by 8th. 1. of Elr.) those triangles are equiangular, and ∴ the angle ABC is bisected by BD; also the angles ABE and EAB are together = to CBE and ECB; ∴ the angle AED (which is = to one pair) is = to CED (that is, = to the other pair) ∴ AEC is bisected; ∴ AED is double of ABE; ∴ the angles ABE and BAE are =; in like manner it can be proved, that CBE and ECB are =.

Therefore it is evident, if the angle ABC be bisected by BD, and if the angles BAE and BCE be made each = to half ABC, that the angle AEC will be double of ABC. For it is = to the three angles ABC, BAE and ECB; but they are together double of ABC; ∴ &c.

Note.—This does not hold, when the vertical angle is right.

PROP. 23. THEOR.

If any angle (BAC) *of a triangle be bisected by* (AL) *and a perpendicular* (AE) *be let fall from this angle on the opposite side, the angle* (LAE) *contained by bisecting line and perpendicular, is equal to half the difference between the other two angles of the triangle.*

Fig. 16.

Make AD = to AC; join DC.

Then ADC is an isosceles triangle, and the angle DAC is bisected by AG; ∴ the angle AGD is a right angle (by

Cor. 10. 1. Elr.) and = to AEB; but AGD is = to AKG + GAK (16. 1. Elr.) and AEB is = to ECK + EKC; but EKC is = to AKG; ∴ KAG is = to ECK; but ECK is = to half the difference of the angles ACB and ABC (prop. 18. of dedu.) ∴ &c. &c.

PROP. 24. THEOR.

Right lines (EA *and* CF) *drawn from the extremities of any side* (CE) *of a triangle, to the middle points of the opposite sides, cut one another in a point of trisection.*

Fig. 20. ANALYSIS.

Suppose FI to be one-third of FC, it is ∴ one-half of IC: produce it until IG is = to IC; join EG, and draw DB through the point of intersection I; then the angles IFD and EFG are =, (by 15. 1. Elr.) and the sides about them are respectively = (by constr.) ∴ the angles FEG and FDI are =; but they are alternate; ∴ EG is parallel to BD.

Therefore it is evident, if DB be drawn, and EG parallel to it, and CF produced to G, that FG is = to FI; (26. 1. Elr.) but since CB is = to BE, and BI par. to EG, CI is = to IG; (prop. 6. of Anal.) ∴ IF is one-half IC, and ∴ one-third of CF. In like manner it can be proved that AI is one-third of AE.

PROP. 25. THEOR.

If lines (CF *and* EA) *be drawn from any two angles of a triangle to bisect the opposite sides, the line* (DB) *drawn from the other angle, through their point of intersection,* (I) *bisects the third side.*

Fig. 20.

Produce CF until FG is = to IF and join GE.

Then it is evident from prop. 24. of Anal. that CI is = to IG, and that GE is parallel to IB; ∴ (by prop. 6. of Anal.) CB is = to CE.

PROP. 26. THEOR.

If a right line (BD) *bisect the vertical angle and base of a triangle, it is isosceles.*

Fig. 21.

Produce BD until DE is = to it; join EC.

Then in the triangles BDA and EDC, the angles ADB and CDE are =, and the sides about them are respectively =; ∴ CE is = to BA, and the angle ABD is = to the angle DEC; ∴ DEC is = to CBD; ∴ EC is = to CB; (prop. 6. 1. Elr.) but EC is = to AB (by dem.); ∴ AB is = to CB; ∴ the triangle is isosceles.

PROP. 27. PROB.

To subtend a given rectilineal angle (BAC) *with a given right line, making an angle equal to a given one.*

Fig. 22.

ANALYSIS.

Suppose it done, and that BC is = to the given line, and the angle BCA = to the given one, at which BC is to be drawn; draw CD = and parallel to BA and join AD. Then it is evident, that AD is = and parallel to BC, and that the angle BCA is = to CAD.

Therefore, if from the vertex A of the angle to be subtended, AD is drawn = to the given line, and making the angle CAD = to the angle at which the line is to be drawn; it is evident, if DC be drawn par. to AB, and CB par. to DA, that CB is = to the given line, and ACB = to the given angle.

PROP. 28. PROB.

Given, of a right angled triangle, the hypotenuse (BC) *and point* (D) *where perpendicular falls, to construct it.*

Fig. 23.

ANALYSIS.

Suppose BGC to be the required triangle. Bisect BC in E; join EG; it is = to EB or EC.

Therefore, if you raise a perpendicular DG from the given point and bisect BC in E, and subtend the angle BDG with EG drawn from the point of bisection and join BG and CG: BGC is the required triangle. For the angle BGE is = to GBC, and also the angle CGE to ECG; ∴ the angle BGC is = to the two GBC and GCB, and ∴ CGB is a right angle (cor. 1. 32. 1. Elr.).

PROP. 29. PROB.

Given the perpendicular of an equilateral triangle to construct it.

Fig. 24.

Draw any indefinite right line AB through the extremity of the given perpendicular CD, and at right angles to it; through the other extremity D draw DE and DF making the angles EDC and FDC each = to one-half the angle of an equilateral triangle; DEF is the required triangle.

For since CDF is an angle of 30 degrees, and DCF an angle of 90°, CFD must be an angle of 60°; ∴ the angle of an equilateral triangle; ∴ DEF must be an angle of an equilateral triangle.

PROP. 30. THEOR.

The rectangle under the hypotenuse and perpendicular of a right angled triangle, is equal to the rectangle under the sides.

Fig. 25.

For, complete the rect. DC under the hypot. and perpendicular, and also the rect. BF under the sides: then it is evident, that the given triangle ABC is one-half of each; ∴ they are =.

PROP. 31. PROB.

Given (BC) *equal to the sum of one side and hypotenuse of an isosceles right angled triangle, to construct it.*

Fig. 26.

ANALYSIS.

Suppose BAD to be the required triangle. Join CA. Then because the triangle is isosceles and the angle BAD a right angle, each of the angles DBA and ADB is one-half a right angle: and, since DC is = to DA, the angles DCA and DAC are =, and each of them one-fourth of a right angle, since they are together = to ADB.

Therefore it is evident, that, if from either extremity B, of the given sum, BA be drawn, making the angle CBA = to one-half a right angle; and if from the other extremity C, CA be drawn, making the angle BCA = one-fourth of a right angle, and if through A, AD be drawn making CAD = DCA, BAD is the triangle required.

PROP. 32. THEOR.

If the three sides of any triangle (AGC) *be bisected, the perpendiculars drawn to the sides from the points of bisection, meet in the same point. And the point of concourse is equally distant from the three angles.*

Fig. 27.

Draw EI and FI from the points of bisection E and F at right angles to GC and CA: the line drawn from I to B will be at right angles to AG.

For, join IA, IC and IG; because GE is = to EC and IE common and at right angles to GC; GI is = to IC: for the same reason IC is = to IA; ∴ IG is = to IA. Then because IG is = to IA, the angles IGB and IAB are = and AB is = to GB (by hypot.) ∴ the angles GBI and ABI are = and are ∴ right.

PROP. 33. PROB.

To determine a point in the side of any triangle, (ABC) *from which lines, drawn parallel to the other sides of the triangle, will form an equilateral parallelogram.*

Fig. 28. ANALYSIS.

Suppose it done, and that IE is an equilateral parallelogram; join AD. Because IA is = to ID, the angle IAD is = to IDA; for the same reason the angle EAD is = to EDA; but the angle EDA is = to DAI; ∴ the angle EAD is = to IAD.

Therefore, it is evident, that if the angle BAC be bisected, and the line bisecting it be produced to the opposite side, the point D, where it meets it, is the required point. For draw DE and DI par. to BA and AC; the parallelogram IE is equilateral. Because the angle IDA is = to DAE, and IDA is also = to ADE; ∴ IAD is = to IDA: ∴ IA is = to ID; ∴ &c.

PROP. 34. PROB.

Given the sum of the three sides, (AB) *required to construct a triangle, equiangular to a given one.*

Fig. 29.

ANALYSIS.

Let the given sum be AB; and suppose the triangle DEF to be the required one, and that DE is = to DA, and FE to FB, draw AG par. to DE, and BG par. to FE; the triangle AGB is equiangular to DEF; join AE and EB; the angle EAG is = to AED; and AED is = to EAD; ∴ the angles DAE and GAE are =; for the same reason the angles EBG and EBF are =.

Therefore, it is evident, that if you construct on the given sum AB, a triangle AGB equiangular to the given one, bisect the angles GAB and GBA by the right lines EA and EB, and draw through the point E, in which they meet, lines ED and EF par. to GA and GB; the triangle DEF is the required one.

It is evidently equiangular to the given one: and, because ED is par. to GA, the angles GAE and AED are =; but GAE is = to EAD (by construction); ∴ the angles DAE and DEA are =; ∴ DE is = to DA; for the same reason FB is = to FE; ∴ &c. &c.

PROP. 35. THEOR.

If any point (G) *be taken in the side of an equilateral triangle*, (DEF) *and from it lines* (GI *and* GK) *be drawn parallel to the other sides of the triangle, their sum is equal to a side of the triangle.*

Fig. 24.

Because GI is par. to DF, the angle GIE is = to the angle DFE; ∴ GEI is equilateral; ∴ GI is = to GE; for the same reason GK is = to DG; ∴ GK and GI are together = to DE.

PROP. 36. THEOR.

If a side, (CA) *about the bisected angle* (CAB) *of any triangle, be produced, until the produced part* (AK) *is equal to the conterminous side,* (AB) *the right line* (KB) *joining their extremities, will be parallel to the bisector* (AD).

Fig. 28.

Because AB is = to AK, the angle ABK is = to the angle AKB; ∴ ABK is half of their sum; and CAB is = to their sum: ∴ half CAB (viz. DAB) is = to ABK; but they are alternate; ∴ KB is parallel to AD.

PROP. 37. PROB.

To draw a right line parallel to the side (AC) *of any triangle*, (ABC) *so that the alternate segments* (AD *and* BE) *shall be equal.*

Fig. 30. ANALYSIS.

Suppose it done, and that DE is drawn par. to AC, making the alternate segments AD and BE =, draw EK par. to BA and join KB.

Because EK is par. to DA, and ED to KA. EK is = to DA; ∴ to EB; ∴ the angles EBK and EKB are =; but EKB is = to KBA; ∴ KBA is = to EBK.

Therefore, if the angle ABC be bisected by BK, and KE drawn parallel to AB, the right line drawn from E, parallel to AC, makes the alternate segments, i. e. BE and DA, =. For EK is = to DA, and BE is = to EK, (because the angle BKE, as being = to ABK, is = to KBE) ∴ BE is = to DA.

PROP. 38. PROB.

Given one side, (AB) *an angle at it, and the difference between the other two sides to construct the triangle.*

Fig. 31. ANALYSIS.

Suppose ADB to be the required triangle, when the given angle DBA is subtended by the lesser side AD. Cut from DB (the greater) CB = to the given difference; join CA. Then AD is = to CD; ∴ the angle DAC is = to DCA (5. 1. Elr.).

Therefore it is evident, if you draw CB = to the given difference, and making with AB the angle ABC = to the given angle, join CA, produce BC, and make the angle CAD = to the angle ACD, that ADB is the required triangle.

Fig. 32.

But, when the given angle ADB is subtended by the greater side AB, produce BD until CD is = to the given difference; join AC. Then, since BA is = to BC, the angle BAC is = to BCA.

Therefore, if you make the angle ADB = to the given angle, make DC = to the given difference; join CA, and make the angle CAB = to BCA, it is evident that ADB is the required triangle.

PROP. 39. PROB.

Given the base (AB) *to find the locus of the vertex of an infinite number of triangles equal in area.*

Fig. 33. ANALYSIS.

Suppose any number of triangles ACB, ADB, and AEB = in area, to be on the given base ; join C and D the vertices of any two of them ; since the triangles are =, and on the same base, they must be between the same parallels ; ∴ CD is par. to AB, and for the same reason DE is par. to AB ; ∴ it is evident, that the locus lies in a line par. to the base.

PROP. 40. THEOR.

The right lines (DE, EF, FG and GD) *joining the middle points of the sides of any quadrilateral figure, form a parallelogram, which is half of the entire figure* (ABCI.)

Fig. 34.

Draw IB and AC. Then in the triangle AIC, since the sides AI and IC are bisected, GD connecting the points of section is par. to AC (prop. 7 of anal.) ; for the same reason FE is parallel to AC ; ∴ FE is par. to GD ; in like manner it can be proved, that DE is par. to GF ; ∴ FEDG is a parallelogram.

And because AI and IC are bisected in G and D, the triangle GID is the fourth part of AIC ; also FBE is the fourth part of ABC ; ∴ the triangles GID and FBE are together the fourth part of the entire figure ; in like manner it can be proved, that the triangles DCE and GAF, are together the fourth of the entire figure ; ∴ if all those triangles be taken away, the remaining figure GDEF is half of the entire figure.

PROP. 41. THEOR.

If in any triangle (ABC) *a line* (BO) *be let fall from the vertex perpendicular to the base, and if, from the point in which the perpendicular meets the base, lines* (OD and OE) *be drawn to the points of bisection of the other sides, the triangle* (DOE) *formed by these lines and the line* (ED) *joining the points of bisection, is similar to the whole triangle.*

Fig. 35.

Because the side BC is bisected by DI, BO is also bisected by it, and the angles BID and DIO are right; ∴ BD is = to DO; but BD is = to DC; ∴ DO is = to DC; ∴ the angles DCO and DOC are =; but the angle EDO is = to DOC, ∴ to DCO; for the same reason the angle DEO is = to CAB; ∴ the remaining angle DOE is = to the angle ABC; ∴ the triangles BAC and OED are similar.

Cor. It is also evident, that, if AC be bisected in G, the triangle DGE is similar to ABC.

PROP. 42. PROB.

To inscribe a square in an equilateral triangle (ABG).

Fig. 36. ANALYSIS.

Suppose CF to be the required $\square^2$, let fall a perpendicular BD; join BE. Then LF is = to FB, since the triangle LBF is evidently equilateral; ∴ FB is = to FE; ∴ the angle FEB is = to FBE; but, since FE is parallel to BD, the angle EBD is = to FEB, and ∴ to FBE.

Therefore it is evident, if the perpendicular BD be drawn, and the angle DBG be bisected by BE, that the perpendicular, raised from the point E to meet BG, is the side of the required $\square^2$. For the angle FEB is = to EBD, and ∴ to EBF; ∴ FE is = to FB, and ∴ to FL; ∴ &c. &c.

PROP. 43. PROB.

Given (AB) *the difference between the hypotenuse and side of an isosceles right angled triangle to construct it.*

Fig. 37. ANALYSIS.

Suppose ACD to be the required triangle; join CB.

Then, because AB is the difference, DB is = to DC; ∴ the angle DCB is = to DBC; but these angles are together the supplement of half a right angle; ∴ DBC is half the supplement of half a right angle.

Therefore, if at one extremity A of the given dif. you make BAC = to half a right angle, and at the other extremity B, make the angle DBC = to half the supplement of half a right angle; then produce AC and BC to meet, and draw CD at right angles to AC; the triangle ACD is evidently the required one.

PROP. 44. THEOR.

If two triangles (BAC and CAD) *have two sides* (BA and AC) *of the one respectively equal to two sides* (CA and AD) *of the other, and the contained angles* (BAC and CAD) *supplemental, the two triangles are equal.*

Fig. 38.

For, let the = sides BA and AD be placed *in directum*, the other two = sides must coincide, since the angles are supplemental; ∴ BAC and CAD are of the same altitude, and, being on = bases, are = to one another.

PROP. 45. PROB.

To cut from a given line (AB) *a part, whose square shall be equal to half the square of the given line.*

Fig. 39.

ANALYSIS.

Suppose the $\square^2$ of CB to be = to half the $\square^2$ AF formed on the given line; bisect AB in G, and draw GE parallel to BF or AD; GEFB is half the $\square^2$ of AB, and is ∴ = to the $\square^2$ of CB; but GF is also = to the $\square^2$ of HB, the side of an isosceles right angled triangle, formed on the given line; ∴ CB is = to HB.

PROP. 46. THEOR.

If any two points (B and C) *be taken in the base* (DE) *of an isosceles triangle* (DAE), *the perpendiculars* (BF and BG) *from one point, are equal to those* (CI and CH) *from the other point.*

Fig. 40.

For, draw BK parallel to DA, and CL par. to EA, it is evident, that, of the two BF and BG, BF is = to KI, and LG to CH (of the other two CI and CH), and because BK is par. to DA, and CL to AE, the angles BKC and CLB are = to one another; for the same reason, the angles KBC and LCB are =, and BC common to the triangles BKC and CLB; ∴ BL is = to BK (by 26. 1. Elr.); ∴ CI and CH are together = to BA and BG.

PROP. 47. THEOR.

Of all the lines, that can be drawn from any point (A) *to a given right line* (BC), *a perpendicular is the least; and, of all the others, that, which is nearest to the perpendicular, is less than the one more remote.*

Fig. 41.

For, if it be possible, let AF be drawn less than the perpendicular AD; then, because AF is less than AD, the angle AFD is greater than ADF, i. e. an acute angle, greater than a right, which is absurd; ∴ AF is not less than AD. Also, AB is greater than AF, because it subtends a greater angle.

PROP. 48. PROB.

To divide a given right line, (AB) *so that the square of one part shall be equal to twice the square of the other.*

Fig. 42. ANALYSIS.

Suppose that the $\square^2$ of AC is = to twice the $\square^2$ of CB, it is evident, that CB must be the side of an isosceles right angled triangle, of which AC is the hypotenuse. Describe on AC an isosceles right angled triangle CDA; join DB. Then each of the angles DAC and DCA is half a right angle, and DCA is = to the sum of the angles CBD and DCB; but those angles are = to one another, because CA and CD are =; ∴ each of them is the fourth part of a right angle.

Therefore it is evident, if, at either extremity A, the angle BAD is made = to half a right angle, and from the other extremity B, BD be drawn to meet it, making the angle ABD = to one-fourth of a right angle, and if D be drawn, making the angle BDC = to the angle CBD, that C is the required point.

PROP. 49. PROB.

Given the periphery (BE) *of an isosceles right angled triangle to construct it.*

Fig. 43. ANALYSIS.

Suppose it done, and that ADC is the required triangle ; join AB and AE ; then the angle DCA is = to half a right angle ; ∴ because CA is = to CB, the angles CAB and CBA are =, and each one-fourth of a right angle ; for the same reason, each of the angles AED and EAD is one-fourth of a right angle.

Therefore, draw AE and AB, making each of the angles AEB and ABE = to one-fourth of a right angle ; and also draw DA and CA, making the angles DAE and CAB = to the angles AEB and ABE ; it is evident, that the triangle ADC is the required one.

PROP. 50. THEOR.

The angle (BAD) *of a regular pentagon* (EBADC) *is trisected by the diagonals* (EA and CA).

Fig. 45.

Because the sides EB and BA are = to CD and DA, and the contained angles =, the angles BAE and DAC are = ; also the angles DAE and CEA are = ; and the angles ECD and ADC are = ; ∴ EA is parallel to CD ; for which reason the angles EAC and ACD are = ; but ACD is = to CAD ; ∴ EAC is = to CAD ; ∴ the entire angle BAD is trisected by the diagonals.

PROP. 51. THEOR.

If, from any point (G) *in the diagonal of a parallelogram* (DABC), *right lines be drawn to the opposite angles, they cut off equal triangles.*

Fig. 22.

Draw the diagonal DB ; then, since the diagonals bisect one another, the triangles DAI and BAI are = ; for the same reason, the triangles BGI and DGI are = ; ∴ the triangles BAG and DAG are =.

PROP. 52. THEOR.

If points (D, E, F, G) *be taken in the sides of a square,* (ABIC) *at equal distances from the angles, the lines connecting them form a square.*

Fig. 44.

In the triangles DGC and GFI, the sides DC and CG are respectively = to GI and IF, and the contained angles are = as being right; ∴ the angle GFI is = to DGC; ∴ the angles DGC and FGI are = to the angles FGI and GFI, and ∴ to a right angle; ∴ DGF is a right angle; for the same reason, the angles GDE, DEF and EFG are right angles; ∴ the figure is rectangular. Also the sides DG and GF are =, and their opposite sides are = to them; ∴, &c.

PROP. 53. THEOR.

If the sides of a regular pentagon be produced to meet in the points (K, L, M, N, O) *all the angles formed at these points are together equal to two right angles.*

Fig. 45.

The angles at those points are evidently = to one another: the angle DAB is six-fifths of a right angle; ∴ each of the angles KAD and KDA is four-fifths of a right angle; ∴ each of the angles K,L,M,N,O, is two-fifths of a right angle; if then they be added together, they are = to two right angles.

PROP. 54. PROB.

Given in any triangle the perpendicular let fall from the vertical angle on the base, and the difference between each segment made by the perpendicular, and its adjacent side, to construct the triangle.

Fig. 37.

ANALYSIS.

Suppose ACD to be the required triangle, and IB to be the difference between the segment DI and its conterminous side DC, and IF the difference between the seg. AI and its conterminous side AC. Because AC is = to AF, the angles ACF and AFC are =; for the same reason the angles DCB and DBC are =.

Therefore it is evident, if you erect a perpendicular IC, = to the given one, from any point in an indefinite right line, make IF and IB = to the given differences, join BC and FC, make the angle BCD = to CBD, and the angle FCA = to CFA, that ACD is the required triangle.

PROP. 55. THEOR.

No two lines from any points (D and E) *in the sides of a triangle, drawn to the opposite angle, can mutually bisect one another.*

Fig. 46.

If one of them, BE, be bisected in F, the other, CD, is not; for, if it were, then, in the triangles DFB and CFE, the sides DF and FB would be respectively = to CF and FE, and the angle CFE being = to the angle DFB, the angle FDB would be = to FCE; but those angles are alternate; ∴ AB would be parallel to AC, which is absurd.

PROP. 56. THEOR.

If any point (H) *be taken, outside a parallelogram* (AF), *and lines be drawn from this point to each extremity of the sides of the parallelogram, the difference between the triangle* (HAB) *formed by one pair of lines,* (HA and HB) *and the triangle* (HDF) *formed by the other pair of lines,* (HD and HF) *is equal to half the given parallelogram.*

Fig. 39.

Draw through the given point LK parallel to AB, and produce DA and FB to meet it.

Then the triangle HAB is half the parallelogram KB, and the triangle HDF is half the parallelogram KF: but the difference between the parallelograms KB and KF is AF; ∴ the difference between their halves (or between HAB and HDF) is half AF.

PROP. 57. PROB.

Two right lines (AB and CB) *being given in position, it is required to place a line between them equal to a given right line, and forming with them an isosceles triangle.*

Fig. 47.

Draw, through the point B, any line, EF, making = angles with AB and CB; make BE and BF each = to half the given line; through E draw EA (at right angles to BE) to meet BA; and through F draw FC parallel to EA ; the line joining AC is the line required to be drawn.

For, in the triangles BAE and BCF, the angles AEB and ABE of the one are respectively = to CFB and CBF of the other, and the side BE = to BF ; ∴ BA is = to BC (by 26. 1.) ∴ the triangle BAC is isosceles ; and AC is = to EF (by 33. 1.) and ∴ to the given line.

PROP. 58. THEOR.

If two triangles (ABC) *and* (DEF) *be so situated, that the sides* (AB *and* DB) *passing through one extremity of the common base, being produced, make the angle* (F) *equal to* (I) ; *and, the side* (DC) *being produced through the other extremity, if it make the angle* (L) *equal to* (E), *the angle* (X) *shall be double of the angle* (D).

Fig. 48.

For L is = to the sum of D and I ; ∴ E is = to the sum of D and I ; ∴ D is the difference between I and E ; also O is = to the sum of D and E ; ∴ twice D is the difference between O and I ; but I is = to K, and O is = to X and K, ∴ X is double of D.

PROP. 59. THEOR.

If any triangle, (ABD,) *if the angles* (ABD and ADC) *be bisected, and the lines bisecting them be produced till they meet* (*in* G,) *the angle* (BGD) *is half* (BAD).

Fig. 41.

For the whole angle ADC is = to the sum of ABD and BAD (prop. 16. 1. Elr.) ; ∴ half of ADC, (viz. GDC,) is = to half ABD, (viz. GBD,) and half BAD ; but GDC is equal to the sum of DBG and BGD ; ∴ BGD is = to half BAD.

PROP. 60. THEOR.

If an angle (ABC) *at the base of an isosceles triangle be bisected, and the right line* (CD) *bisecting it be produced to meet a right line* (AD) *parallel to the base, it makes with it an angle equal to half of one of the base angles.*

Fig. 49.

Produce the side BA through the vertex.

Then the angle EAC is evidently bisected by AD; ∴ (by prop. 59.) the angle ADB is half of the angle ACB; but half ACB is = to ABD; ∴ &c. &c.

PROP. 61. PROB.

In proposition fifty-nine of these deducibles, in what case will the right line (AD) *connecting the vertices be parallel to the base.*

Fig. 49. ANALYSIS.

Suppose AD is parallel to BG; then the angle ADB is = to the angle DBC, and ∴ to DBA; ∴ AD is = to AB: also the angle ADC is = to DCG, and ∴ to ACD; ∴ AD is = to AC; ∴ the triangle ABC will in this case be isosceles.

PROP. 62. THEOR.

If a right line (AD) *be drawn bisecting an angle* (ABC) *at the base of an isosceles triangle, to meet a line parallel to the base, the right line* (DC) *drawn from its extremity to the external angle* (ACG) *bisects it.*

Fig. 49.

For the angle ADB is = to DBC, and ∴ to ABD; ∴ AD is = to AB; ∴ AC is = to AD, and the angle ADC to the angle ACD; but ADC is = to DCG; ∴ DCG is = to DCA; ∴ &c.

Cor. It immediately appears from this, that, in the case of an isosceles triangle, AD is par. to BG, if the angles are bisected by BD and CD. For, if not, draw AI par. to it; join IC. IC bisects ACG; but DC also bisects it, which is absurd.

PROP. 63. PROB.

In proposition fifty-nine, when will the figure (ABCD) *be a parallelogram.*

Fig. 49. ANALYSIS.

Because ABCD is a parallelogram, AD is = to BC (by prop. 34. 1. Elr.); it is also = to AC and to AB, (by the proof of foregoing prop.); ∴ AB, BC and CA are = to one another; ∴ the triangle ABC is equilateral.

If the triangle ABC be equilateral, ABCD is a parallelogram.

Since the angle ACG is bisected, DCG is an angle of an equilateral triangle; but DCG is = to the sum of DBC and BDC; and DBC is half an angle of an equilat.; ∴ CDB is half an angle of an equilat.; ∴ BC is = to CD; and in the triangles CBD and ABD, the sides CD and BD are respectively = to AB and BD, and the angle CBD = to ABD; ∴ CD is = to DA; ∴ BC, CD, DA and AB are = to one another; ∴ the figure is a parallelogram.

PROP. 64. PROB.

Given the base of any triangle, the area, and the line bisecting the base, to construct it.

Divide the given area by half the base and you have the altitude; erect from the middle point of the base a perpendicular = to the altitude, and draw through its extremity a line parallel to the base; inflect from the middle point of the base to this parallel, a line = to the given line bisecting the base, and connect its extremity with the extremities of the base; thus the required triangle is formed; which is evident from prop. 41. 1. Elr.

PROP. 65. THEOR.

In a right angled triangle (CBA), *if one angle at the hypotenuse be double of the other, the hypotenuse is double of the side subtending the lesser angle.*

Fig. 8.

Bisect CA in D; join DB.

Because BCA is double of BAC, BCA is an angle of 60 degrees; and because CBA is a right angled triangle, DB is = to DC (by prop. 8. dedu.); ∴ DBC is an angle of 60°, and ∴ CDB; ∴ the triangle ACD is equilateral; ∴ BC is = to CD, and ∴ half of CA.

PROP. 66. THEOR.

In an equilateral triangle (BAC), *if* (BD) *be perpendicular to* (AC), *and* (DO) *to* (CB), *and* (IO) *equal to* (IB), *the triangle* (DOI) *is equilateral.*

Fig. 21.

Because DOB is a right angled triangle, having the angle OBD an angle of 30 degrees, the angle BDO must be an angle of 60°; ∴ it is evident (from prop. 60 of dedu.) that, if IO be = to IB, the triangle IOD is equilateral.

PROP. 67. PROB.

To determine a point within a triangle, from which the perpendiculars let fall on the sides shall be equal.

Fig. 50.

ANALYSIS.

Because DE is = to DF and DB common, and the angle DEB = to DFB, and the angles EDB and FDB of the same affection, the angle EBD is = to FBD; for the same reason the angles FCD and GCD are =, and also GAD and EAD.

Therefore, if you bisect each of those angles, the point D, in which the lines bisecting them meet, is the required point. For in the triangles DBF and DBE the side DB is common, and the angles DFB and DBF of the one respectively = to DEB and DBE of the other; ∴ DF is = to DE: for the same reason DF is = to DG; ∴ &c. &c.

PROP. 68. THEOR.

If two sides (CD and EF) *of a quadrilateral figure be parallel, the triangle* (ICE) *contained by either of the other sides, and the right lines drawn from its extremities to the point of bisection of the opposite side, is half the figure.*

Fig. 51.

Draw GH through the point of bisection I, parallel to CE, and draw FH. Then the triangle GID is evidently = to HIF; ∴ the parallelogram CH is = to the given figure ED; but the triangle CIE is = to half the parallelogram, and ∴ to half the figure.

PROP. 69. PROB.

To bisect a given quadrilateral figure (GE) *by a right line drawn from one of its angles* (C).

Fig. 52.

Draw diagonals CD and GE, and bisect the diagonal GE, opposite the angle C, in O; through O draw LI parallel to CD; the right line drawn from C to I bisects the given figure; join CO and DO.

Because GO is = to OE, the triangles GOC and EOC are =; for the same reason the triangles DOG and EOD are =; ∴ the figures GCOD and CODE are =; and the triangles COD and CID are =; ∴, if the common part OFI be taken away, the remainders COF and FID are =; ∴ the triangle CIE is = to the figure CIDG.

PROP. 70. THEOR.

If two sides (CE and EB) *of any parallelogram be bisected* (in G and I), *and lines drawn from the points of bisection to the opposite angle, they trisect the diagonal* (CB) *subtending this angle.*

Fig. 15.

For, in the triangle DCE, since the sides DE and EC are bisected in A and G, the lines CD and AC drawn to the opposite angles, cut one another in a point of trisection (prop. 24 of dedu.); ∴ AP is the third part of AC; it is ∴ half of PC; for the same reason AO is half of OB; but AB is = to AC, and PO is the third part of their sum; ∴ it is evident, that OB and PC are each = to PO; ∴ &c.

PROP. 71. THEOR.

If in a triangle (ABC) *a line* (BD) *drawn from the vertical angle to the point of bisection of the base, be greater than half the base, the vertical angle is acute; and if the line be less than half the base, it is obtuse, as in the triangle* AEC.

Fig. 19.

Because DB is greater than DA, the angle DAB is greater than DBA; for the same reason the angle DCB is greater than DBC; ∴ the sum of the two angles BAC and BCA are greater than ABC; ∴ ABC is less than a right angle. Thus it can be proved, that AEC is greater than a right angle, since DE is less than DA.

PROP. 72. PROB.

Given in a right angled triangle, one side and the sum of the hypotenuse and the other side, to construct it.

Fig. 53. ANALYSIS.

Suppose ABC to be the required triangle: AB is = to the given side, and AD the given sum; join DB. Since BAC is a right angle, BC is the hypotenuse, and it is = to CD, (the difference between the second side AC and the given sum); ∴ the angles CBD and CDB are =.

Therefore it is evident, if you form with the given side BA and given sum AD, a right angle BAD, join DB, and make the angle DBC = to the angle CDB, that BAC is the required triangle.

PROP. 73. PROB.

Given in any triangle the base, the difference of the sides, and the difference of the base angles to construct it.

Fig. 31. ANALYSIS.

Suppose ADB to be the required triangle. Let AB be = to the given base, and BC = the given difference. Join CA. Because CB is the difference between AD and DB, AD is = to DC; ∴ the angles ACD and CAD are =: also because CB is the difference, the angle CAB is = to half the difference of the base angles.

Therefore, it is evident, if from one extremity (A) of the given base, you draw AC, making the angle CAB = to half the difference of the base angles, from the other extremity B, inflect BC = to the difference of the sides, produce BC, and through A draw AD, (to meet it,) making the angle CAD = to ACD, that ADB is the required triangle.

PROP. 74. PROB.

To bisect a given parallelogram (EA) *by a right line drawn from a given point* (O) *without it.*

Fig. 4.

Bisect the diagonal AE, and draw OL through the point of bisection; it bisects the parallelogram.

For the triangles DAB and BED are =; also the triangles DGK and BGL are = ; ∴ the figure EKLB is = to the figure LADK.

PROP. 75. PROB.

To find two equal squares, whose sum shall be equal to the sum of two unequal squares.

Fig. 54.

With the sides (GK and GL) of the unequal squares form a right angle KGL ; draw the hypotenuse KL, and on it describe an isosceles right angled triangle KML ; the sum of the squares of KM and LM is = to the sum of the squares of KG and GL.

For the squares of KG and GL are = to the square of KL (prop. 47. 1. Elr.) ; also the squares KM and ML are = to the square of KL ; ∴ &c.

PROP. 76. THEOR.

It is required to draw from two given points (A *and* B) *to the same point in a right line* (GD) *given in position, two right lines, whose difference shall be a maximum.*

Fig. 31.

Draw from A the right line AI at right angles to DG, and produce it until IC is = to IA ; join BC, and produce it to D ; join DA.

Then it is evident, that CB is the difference between AD and DB : it is greater than the difference between any other lines (AO and OB) drawn to any point O in DG ; join OC. Then AO is = to OC ; but OB is less than the sum of OC and CB ; ∴ the difference between BO and OC is less than CB ; ∴ the difference between AO and OB is less than CB ; ∴ &c.

PROP. 77. THEOR.

If two right angled triangles (ABC and DEF) *having equal hypotenuses,* (AC and DF,) *have two other sides also equal, the remaining sides shall be equal, and the two triangles shall be equal in every respect.*

Fig. 55.

Produce AB until BG is = to ED ; join GC.

Then the triangles BCG and DEF have BG = ED, and BC = EF, and the angle CBG = FED; they will also have the angle G = D, and GC = DF = AC: whence the triangle ACG being isosceles, the angles G or D will be = A, and consequently the angle F = ACB; ∴ the triangles ABC and DEF being equiangular, and having AC = DF, they are = in every respect.

PROP. 78. THEOR.

If two triangles (ABC *and* DEF) *have two sides* (AC *and* BC) *of the one respectively equal to two sides* (DF *and* EF) *of the other, and have the angle* (CAB) *subtended by the side* (CB) *equal to the angle* (FDE) *subtended by the side equal to it* (FE)*; and if the angles* (CBA *and* FED) *subtended by the other equal sides, be of the same affection, i. e. both acute or both obtuse; the two triangles will be equal in every respect.*

Fig. 56.

Draw CG and FH at right angles to AB and DE.

Then the angle AGC being = to DHF, and the angle A = D, and the side AC = DF, CG is = to FH; whence CB being = to FE, the angles GBC and HEF are also =, (prop. 77 of dedu.); and then the triangles ABC and DEF being mutually equiangular, and having the sides AC and DF =, they are = in every respect.

If the angles be both obtuse, as in the triangles AIC and DLF, the demonstration is the same:

For, if CI (= CB = FE) = FL, the angles GIC and HLF being = (as before), the angles AIC and DLF shall be =.

PROP. 79. THEOR.

If the angles at the base (ML) *of an equilateral triangle be bisected by the right lines,* (MI *and* LI,) *the lines* (IR *and* IS) *drawn from their point of concourse* (I) *parallel to the other two sides,* (MO *and* LO) *trisect the base.*

Fig. 57.

Because IR is parallel to MO, the angle RIM is = to IMO, and ∴ to IMR; ∴ MR is = to RI; for the same reason LS is = to SI: again, because RI is parallel to MO, the angle IRS is = to OML; for the same reason

ISR is = to OLM ; ∴ the triangle IRS is equilateral ; ∴ IR and IS are each = to RS ; ∴ MR, RS, and SL, are = to one another.

PROP. 80. THEOR.

If from one extremity (A) *of the base of an isosceles triangle* (DAC) *any right line* (AK) *be drawn to the opposite side, and from the same extremity another right line* (AB) *be drawn to the same side produced, so that the intercept* (KB), *between the two drawn lines, shall be equal to the first drawn line, the angle* (KAD), *contained by the first drawn line and the other side of the triangle, is double the angle* (CAB) *contained by the second drawn line and base.*

Fig. 31.

For the angle KCA is = to CBA and CAB ; ∴ it is = to KAB and CAB : also the angle DAC, since it is = to DCA, is = to the sum of KAB and CAB, i. e. to KAC and twice CAB ; ∴ if KAC be taken away, DAK will remain = to twice CAB.

PROP. 81. THEOR.

The sum of the sides of an isosceles triangle (ABC) *is less than the sum of the sides of any other triangle* (DBC) *on the same base* (BC) *and between the same parallels.*

Fig. 49.

For, since FD is parallel to BC, the angles FAB and ABC are = ; also the angles DAC and ABC ; ∴ the angles DAC and FAB are = ; ∴ the sum of the right lines BA and CA is less than the sum of any other two lines BD and CD, that can be drawn from the points B and C to FD. (prop. 10 of deducibles.)

PROP. 82. THEOR.

If the external angles (OKL *and* KLR) *of any triangle* (KML) *be bisected by the right lines,* (KN *and* LN,) *the right line* (NM), *drawn from the point of concourse* (N) *to the opposite angle, bisects it.*

Fig. 58. ANALYSIS.

Suppose the angles OMN and RMN are =, draw the perpendiculars NR and NO: they are evidently =, (by prop. 26. 1. Elr.); if this can be proved, the proposition is true.

Draw NI at right angles to KL. Because the angles NLR and NLI are =, and also the angles NIL and NRL, and the side NL common to the triangles NIL and NRL, NI is = to NR; for the same reason NI is = to NO; ∴ NO is = to NR. Then, since NR is = to NO, and NM common to the triangles NMR and NMO, and the angles NRM and NOM right angles, the angles NMR and NMO must be of the same affection; ∴ the triangles are = in every respect, and ∴ the angles NMR and NMO are =.

PROP. 83. PROB.

To draw from the extremity (A) *of one side* (AB) *of a parallelogram to the opposite side,* (KB,) *a right line which shall be equal to the side, to which it is drawn, together with the segment of it intercepted between the drawn line and adjacent angle* (K.)

Fig. 59. ANALYSIS.

Suppose it done, and that AP is = to the sum of BK and KP. Produce PK till PO is = to PA; join OA. Because PO is = to PA, PO is = to the sum of BK and KP; ∴ KO is = to KB; also the angles POA and PAO are =.

Therefore it is evident, if you make KO = to KB, join OA, and make the angle OAP = to POA, that AP is the line required to be drawn.

PROP. 84. THEOR.

The area of a trapezium (LS) *is half that of a parallelogram* (LB) *whose base* (AB) *is equal to the sum of the parallel sides* (LE and AS) *of the trapezium, and whose altitude is equal to that of the trapezium.*

Fig. 59.

Draw ET and RS parallel to LA or KB.

Then, since LE is = to SB, AS is = to TB; ∴ the parallelograms RA and EB are =; also the triangle

ETS is = to SRE; ∴ the trapezium LASE is = to ESBK; ∴ &c. &c.

PROP. 85. THEOR.

If perpendiculars (PI, CI, DI) *be drawn from the middle points of the sides of an equilateral triangle till they meet in* I, *they are equal to one another; and their sum is equal to the perpendicular* (MC) *drawn from any angle* (M) *to the opposite side.*

Fig. 57.

It is evident, that, if one of the perpendiculars CI be produced, it will bisect the angle OML; then, because DM is = to MP, and MI common, and the angles DMI and PMI =, PI is = to ID; for the same reason PI is = to IC: ∴ they are equal to one another.

And, because CM bisects the angle OML, and also DL bisects the angle OLM, they cut one another in a point of trisection; ∴ CI is the third part of CM; but CI is also the third part of the sum of the three, DI, PI, and CI; ∴ the three together are = to CM.

PROP. 86. THEOR.

In proposition 52, *the internal square is half the external, when the sides of the external square are bisected.*

Fig. 44.

Draw a diagonal BC; then, because the sides of the triangle ACB are bisected in *a* and *b*, the triangle A*ab* is the fourth part of ABC, and ∴ the eighth part of the $\square^2$ AI; for the same reason each of the triangles *a* C *d*, *d* I *c*, and *c* B *b*, is the eighth part of the entire figure; ∴ the four together are = to half AI; ∴ the square *a b c d* is half the square AI.

PROP. 87. PROB.

Given of any triangle the base, and the sum and difference of the angles at the base, to construct it.

Fig. 53.

ANALYSIS.

Suppose ABD to be the required triangle, cut from AD a part AC = to AB; join BC.

Then, because AB is = to AC, it is evident (from prop. 18 of dedu.), that the angle CBD is = to half the difference of the base angles, and also that the angle ABC is = to half the sum; ∴ ABD is = to half the sum + half the difference, and ∴ ADB is = to half the sum − half the difference.

From this it is evident, if, from one extremity B of the given base, you draw BA, making the angle DBA = to half the sum + half the difference, and if, from the other extremity D, you draw DA, making the angle BDA = to half the sum − half the difference, that ABD is the required triangle.

PROP. 88. PROB.

Given in any triangle, one side, an angle at it, and the length of the perpendicular drawn from its middle point, to meet those drawn from the middle points of the other two sides, to construct the triangle.

Fig. 27.

ANALYSIS.

Suppose GAC to be the required triangle. Let GC be the given side, CGA the given angle, and EI the given perpendicular; join IG and IA; bisect GA in B, and draw BI; it is at right angles to GA (prop. 32 of dedu.); and IA is = to IG.

Then at one extremity, G, of the given side, make the angle CGA = to the given angle; draw EI = to the given perpendicular; join IG; and from I draw a line IB perpendicular to GA; produce GB till the production BA is = to GB, and then join AC; AGC is evidently the required triangle.

PROP. 89. PROB.

Given of any triangle, one side and the lines which, drawn from the angles at it, bisect the other two sides, to construct it.

Fig. 20.

ANALYSIS.

Suppose DCE is the required triangle. Let CE be the given side, and CF and EA the given bisectors; it is evident (prop. 24 of dedu.), that they cut one another in a point of trisection; ∴ CI and EI are respectively two-thirds of CF and EA.

Therefore, with the given side CE, and with CI and EI (the respective two-third parts of the given bisectors) form a triangle CIE; produce those parts, until CF and EA are respectively = to the given bisectors; draw through their extremities A and F (from the extremities of the given side), CD and ED, and DCE will be the required triangle.

PROP. 90. PROB.

Given of any triangle, one side, an angle at it, and one of the equal perpendiculars on the sides, to construct the triangle.

Fig. 50.

ANALYSIS.

Suppose ABC to be the required triangle. Let BC be the given side, BCA the given angle, and DF one of the = perpendiculars; join DC, and draw the perpendicular DG. It is evident (prop. 32 of dedu.), that DC bisects the angle BCA.

If, therefore, you bisect the given angle by the right line CD, and subtend the angle BCD with DF = to the given perpendicular, and making with CB the angle CFD a right angle (prop. 27, dedu.); join DB and draw BA, making the angle DBA = to DBC; ABC is evidently the required triangle.

PROP. 91. THEOR.

The square of the hypotenuse (AC) *of an isosceles right angled triangle* (AEC) *is equal to four times the square of the perpendicular* (ED) *from the right angle on the hypotenuse.*

Fig. 19.

For ED is = to AD or DC; but the squares of AE and EC are = to two squares of ED together with the squares of AD and DC; but the squares of AD and DC are = to two squares of ED; ∴ the $\square^2$rs of AE and EC are = to four $\square^2$rs of ED, and ∴ the $\square^2$ of AC, which is = to the $\square^2$rs of AE and EC, is = to four $\square^2$rs of ED.

PROP. 92. PROB.

Given one angle of any triangle, one side, and the length of the line drawn from the point of concourse of the perpendiculars (from the middle points of the sides) to one of the angles, to construct it.

Fig. 27. ANALYSIS.

Suppose that AGC is the required triangle. Let ACG be the given angle, GC the given side, and IC the given line. Draw IF, IE and IB at right angles to the middle points of the sides; join IA and IG. IA and IG are evidently each = to IC (prop. 32 of deduc.)

If ∴ you bisect the given side GC in E, draw EI at right angles to it, draw CI from the given angle = to the given line, and draw (from the point I, where CI meets the perpendicular) IA and IG, each = to CI, to meet the legs of the given angle, and join AG: AGC is evidently the required triangle.

PROP. 93. PROB.

Given of a right angled triangle, the sum of the hypotenuse and one side, and the contained angle to construct it.

Fig. 42. ANALYSIS.

Suppose DCA is the required triangle. Let BA be = to the given sum, and ACD to the given angle; join DB; the angle DBC is half the given angle, because CB is = to CD; and the angle DAC is the complement of the given angle, it with DCA being = to a right angle.

Therefore, if at one extremity A of the given line, you make the angle DAB = to the complement of the given angle, and at the other extremity B make the angle ABD = to half the given angle, and make the angle BDC = to DBC; DCA is evidently the required triangle.

For, the angle DCA is = to CBD + CDB, and ∴ to the given angle; DC + CA are = to AB, because the angles BDC and CBD are =, and the angle CDA is a right angle, because the angles CAD and ACD are together = to a right angle.

PROP. 94. THEOR.

The right line (AD) *drawn from the vertex of any triangle* (ABC) *to bisect the base, is less than half the sum of the sides.*

Fig. 28.

Produce CA till AK is = to it; join KB; then in the triangle CBK, CB is bisected in D, and CK in A; ∴ DA is parallel to BK, and = to half of BK (prop. 7. of deduc.); but KB is less than the sum of BA and AK; ∴ than the sum of BA and AC; ∴ half of BK or AD is less than half of CA and AB.

PROP. 95. PROB.

Given one side, and the perpendicular on the hypotenuse of a right angled triangle, to construct it.

Fig. 25.

Draw any right line AC; from any point O in it, draw a line OB at right angles to it, and = to the given perpendicular; from its extremity B draw the right line BA, subtending the angle BOA and = to the given side, and draw BC, making the angle OBC = to the angle BAC; ∴ BCA is the required triangle.

For, since BOA is a right angle, ABO + BAO are = to a right angle; ∴ OBC + OBA are = to a right angle; ∴ the triangle is right angled, and it has the given side and given altitude.

PROP. 96. THEOR.

If the bisection (ICD) *of the vertical angle of an isosceles triangle* (ACD) *be bisected, the intercept* (IF) *of the base, between the bisectors, is less than the intercept* (FD) *between the side and second bisector.*

Fig. 37.

Draw FG at right angles to CD.

Then, in the triangles CIF and CGF, the angles ICF and GCF are =, also the angles CIF and CGF, and the side CF common to the two triangles; ∴ FI is = to FG (prop. 24. of Elr.); but FG is less than FD, since the angle FDG is less than FGD; ∴ FI is less than FD.

PROP. 97. THEOR.

If in any parallelogram (AF) *right lines be drawn from the opposite angles* (FCA and AEF), *to the points of bisection* (D and G) *of their opposite sides* (AE and CF), *they trisect the diagonal* (AF).

Fig. 47.

Since DE and CG are the halves of opposite sides of a parallelogram, they are =; ∴ CD is parallel to GE. And in the triangle AKE, since AE is bisected in D, and ID parallel to KE, IA is = to IK (prop. 6. of dedu.); for the same reason KF is = to KI; ∴ the three AI, IK and KF are = to one another; ∴ the diagonal is trisected.

PROP. 98. THEOR.

If the right lines (DF and DG) *drawn from the middle point of the base of a triangle* (BAC), *at right angles to the sides, be equal, the triangle is isosceles.*

Fig. 19.

Because, in the triangles FAD and GCD, the sides FD and DA of the one are respectively = to GD and DC of the other; and the angles DFA and DGC righ angles, it follows (by the 47th 1. of Elr.), that FA and GC are =; ∴ by the 8th 1. of Elr. the angles GCD and FAD are =, and ∴ the triangle ABC is isosceles.

PROP. 99. THEOR.

If, from the extremity of the lesser side (AB) *of any triangle* (ABD), *a line* (BC) *be drawn, cutting from the greater a part* (AC) *equal to the lesser, and if the remaining side* (BD) *be bisected* (*in* E) *and a right line* (EF) *be drawn through the point of bisection, parallel to the cutting line, the distance between the point* (F), *where it cuts the greater side, and the vertex of the triangle, is equal to half the sum of the greater and lesser sides, and the remainder is equal to half their difference.*

Fig. 53.

Because the side BD is bisected in E, and EF drawn parallel to BC, CF is = to FD (prop. 6. of deduc.); ∴ BA and FD together are = to AC and CF; ∴ AF is half the sum of BA and AD. And since BA is = to AC, CD is the difference between BA and AD; ∴ FD is half the difference.

PROP. 100. PROB.

Given of any triangle, the base, difference of the sides, and vertical angle, to construct it.

Fig. 53. ANALYSIS.

Suppose that ABD is the required triangle. Let BD be given as its base, CD the given difference, and BAD the given angle; join BC. Because CD is the difference between AB and AD, ∴ AB is = to AC; ∴ the angles ABC and ACB are =, and ∴ ACB is half the supplement of the vertical angle.

If ∴ you draw any line, BC; make, at one extremity C of it, an angle BCA = to half the supplement of the vertical angle; produce the right line AC till CD is = to the given difference; from its extremity D inflect the right line DB = to the given base, and draw BA, making the angle ABC = to ACB; it is evident that ABD is the required triangle.

PROP. 101. PROB.

In a right angled isosceles triangle, given the difference between the hypotenuse and side, to construct it.

Fig. 37. ANALYSIS.

Suppose that CAD is the required triangle. Let AB be = to the given difference; join BC; then DB is = to DC, and ∴ the angle DBC to DCB, and since the triangle CAD is isosceles, the angle CDA is half a right angle; ∴ the angle CBD is half the supplement of half a right angle.

Then at one extremity (A) of the given difference, make the angle BAC = to half a right angle; at the other extremity B, make the external angle DBC = to half the supplement of half a right angle; from the point C, in which the lines making those angles meet, draw CD, making the angle BCD = to CBD; the triangle CAD is evidently the required one.

For, since the angles DCB and DBC are =, and each of them half the supplement of half a right angle, CDB is half a right angle, and is ∴ = to DAC; ∴ ACD must be a right angle; ∴ &c.

PROP. 102. THEOR.

The perpendiculars (DF and DG), *let fall from the middle point of the base of an isosceles triangle on the sides, are equal.*

Fig. 19.

For, in the triangles DFA and DGC, the side DA is = to DC, and the angles DAF and AFD of the one respectively = to DCG and CGD of the other; ∴ the triangles are = in every way, and ∴ DF is = to DG.

PROP. 103. PROB.

To draw a right line parallel to the base (AB) *of any triangle* (GAB), *so that the sum of the intercepts between the parallels, shall be equal to the drawn line.*

Fig. 29.

ANALYSIS.

Suppose that HI is the line required to be drawn. Make IE = to IB; HE is = to HA; join EA and EB. Then, because EI and IB are =, the angles IEB and IBE are =; and because IH is parallel to BA, the angles IEB and EBA are =; ∴ the angles IBE and EBA are =; ∴ GBA is bisected; for the same reason GAB is bisected.

If ∴ the angles at the base be bisected, and the right lines bisecting them be produced to meet, it is evident, that the right line IH, drawn through their point of concourse E, is = to the sum of the intercepts HA and IB.

PROP. 104. THEOR.

If right lines (CE, AG, DH and BF) *be drawn from the angles of any parallelogram* (AD), *at right angles to any line* (EF), *the sum of those* (CE and BF), *from one pair of opposite angles, is equal to the sum of those* (AG and DH) *from the other pair of opposite angles.*

Fig. 60.

Produce CD to M; it is evident, that AL is = to BM;

then through M draw MK parallel to FE; the sum of KE and MF is evidently = to that of PG and OH; through L draw LI parallel to PK; IK is = to LP; ∴ the sum of IK, KE, BM and MF (of one pair of perpendiculars CE and BF), is = to the sum of AL, LP, PG and OH (of the other pair); if ∴ CI is = to DO, the proposition is true. Because IC is parallel to OD, the angles ICL and ODM are =; for the same reason the angles CIL and DOM are =, and CL is evidently = to DM; ∴ CI is = to DO (prop. 26. 1. Elr.); ∴ &c. &c.

PROP. 105. THEOR.

In an equilateral triangle (EFG), *the perpendiculars* (RP and RS), *let fall from any point* (R) *in one side, are together equal to the perpendicular* (FO), *from any angle on its opposite side.*

Fig. 61.

Draw RL parallel to GE; then XO is evidently = to one of the perpendiculars RS; if then PR is = to FX, the proposition is true. Because LR is parallel to EG, the angles FXR and FOG are =; ∴ FXR and FPR are =; for the same reason the angles XRF and PFR are =, and FR common to the triangles PFR and XFR; ∴ FX is = to RP (prop. 26. 1. Elr.); ∴ &c. &c.

PROP. 106. THEOR.

If points (D,F,H,) *be taken in the side* (AB) *of any triangle, at equal distances from one another, and lines* (DE, FG and HI) *be drawn through them parallel to the base, the parts* (EG and GI) *of the other side, intercepted between them, are also equal.*

Fig. 62.

Draw through G, OGL parallel to BA, and produce DE to meet it.

Then the right lines LG and GO are evidently = to one another; and since OL is parallel to HD, the angles ELO and LOI are =; for the same reason the angles LEI and EIO are =; ∴ EG is = to GI.

Cor. From this, it is evident, that if one side of a triangle be divided into any number of equal parts, and if right lines be drawn from the points of section parallel to another side, they divide the third side into the same number of equal parts.

PROP. 107. PROB.

To find a point in the side (AB) *of any triangle* (ABC), *from which a right line may be drawn to the adjacent side* (BC) *equal to the upper segment of the side, from which it is drawn, and making with the side, to which it is drawn, an angle equal to a given rectilineal angle.*

Fig. 63.

ANALYSIS.

Suppose EF is the line required to be drawn; join FA, and draw AG parallel to EF; the angle AGC is = to the given angle; draw through the angle AGC the right line GD parallel to FA to meet BA produced. Then, because GD is parallel to FA, the angles DGA and FAG are =; for the same reason the angles EFA and FAG are =; ∴ the angles AGD and EFA are =; also the angle EAF (= EFA) is = to ADG; ∴ AD is = to AG.

Then, through A draw AG, making the angle AGC = to the given angle; produce BA until AD is = to AG; join DG; through A draw AF parallel to DG; and through F draw FE parallel to GA; FE is the line required to be drawn.

For the angles EFA, FAG, AGD, and ADG are evidently = to one another, and EAF is = to ADG; ∴ to EFA; ∴ EF is = to EA, and the angle EFC is = to AGC, and ∴ to the given angle.

Cor. In the same way a line can be drawn from any side of a triangle to the adjacent side, equal to the upper segment, making an angle with the side, from which it is drawn, equal to a given rectilineal angle.

PROP. 108. THEOR.

If the altitude of any parallelogram (HV) *inscribed in a triangle,* (ABC,) *be half the altitude of the triangle, its base shall be equal to half the base of the triangle.*

Fig. 64.

For, draw AI at right angles to the base BC.

Then, because AI is bisected in U, and UV parallel to BC, AC is also bisected in V; for the same reason CI is bisected in K, and also BI in H, &c. &c.

PROP. 109. THEOR.

Of all the rectangles, that can be inscribed in a triangle, (ABC) *the greatest is that* (LK) *whose base is equal to the semi-base of the given triangle; and whose altitude is equal to its semi-altitude.*

Fig. 64.

For, let DG be any other parallelogram, it is less than LK. Draw AI at right angles to BC, join UH, DH and OH.

LU is evidently = and parallel to BH; ∴ UH is = and parallel to LB; ∴ the parallelograms LI and LH are = to one another; and, since HU is parallel to BA, the triangles LHU and DHU are =; but LHU is half of the parallelogram LI; ∴ DHU is = to half of it; also the triangle OHU is half of the parallelogram OI, and the triangle ODU is half of DU; but both of these triangles are together less than DHU; ∴ their doubles or DI is less than double DHU or LI; for the same reason EI is less than UI; ∴ DG is less than LK.

Cor. It is evident from this, that the greatest parallelogram that can be inscribed in a triangle, is that whose base is the semi-base, and altitude the semi-altitude of the triangle. For, draw UH parallel to AB, it is evident, that the parallelogram UB is = to LK; ∴ &c. &c.

PROP. 110. PROB.

To inscribe a square in any triangle (ABC).

Fig. 65.

ANALYSIS.

Suppose that DG is the required □^{2}; draw the diagonals FE and GD: it is evident, that they bisect one another, and also that each of the angles OFG and OGF is half a right angle; draw any lines HL and IK respectively parallel to FE and GD; the angles LHC and KIB are each = to half a right angle; also draw OB and OC; since those lines bisect the diagonals, they bisect the lines parallel to them, (prop. 14. dedu.).

Then it is evident, if you draw any lines HL and IK, making each of the angles KIB and LHC = to half a right angle, bisect those lines, and, through the points of bisection draw BO and CO, the lines EF and DG drawn through the point, in which they meet, respectively parallel to LG and KI, are evidently the diagonals of the required □^{2}: for, draw GE, ED and DF:

Then, because HL is bisected by CO, EF is also bisected by it, (prop. 14, dedu.); for the same reason DG is bisected by BO; and because DH is parallel to KI, the angle DGF is half a right angle; also EFG is half a right angle; ∴ FOG is a right angle; ∴ each of the angles FOD, DOE, and EOG is a right angle; and since FO is evidently = to GO, the four GO, EO, DO and FO are = to one another; ∴ D F G E is a square.

PROP. 111. THEOR.

If perpendiculars be drawn from three angles of a parallelogram (AD) *to a right line drawn from the fourth angle, the sum of those* (CE and AF) *from the remote angles is equal to that* (DL) *from the adjacent angle.*

Fig. 66.

Draw EG parallel to CD; GD is evidently = to EC; and since EG is = and par. to CD, it is also = and par. to AB; ∴ in the triangles ABF and LEG, the angles ABF and BFA are respectively = to LEG and ELG, and the side EG = to AB; ∴ AF is = to LG; ∴ AF and EC together are = to LD.

PROP. 112. THEOR.

To find a point in one side (AB) *of any triangle, from which you may draw to the adjacent side, a right line equal to half the upper segment of the side, from which it is drawn, and making with this side an angle equal to a given rectilineal angle.*

Fig. 67.

ANALYSIS.

Suppose that GH is the line required to be drawn: produce it till GF is = to GA; join AF; the angles FAG and AFG are =; draw AH, and through B draw BD parallel to GF and produce AH and AF to meet it; BD is evidently = to BA and is bisected in E, and the angle ABD is = to AGF, and ∴ to the given angle.

Therefore, if through B, you draw BD = to BA making the angle ABD = to the given angle, join AD, bisect BD in E, and join EA; it is evident that the line drawn through H, parallel to BE, is the line required to be drawn.

For the angles AGF and ABD are =; also the angles GFA and BDA; ∴ GFA and GAF are =; ∴ GF is = to GA; and since BD is bisected by AE, GF is also bisected by it; ∴ HG is half of GA, and the angle AGF is = to the given angle.

PROP. 113. THEOR.

If the points of bisection (A, B, C, and D) *of any quadrilateral figure (such as* E F G H I) *be connected, the figure thus formed is a parallelogram.*

Fig. 68.

Join EH and FI.

Then, because in the triangle FEH, the sides FE and FH are bisected, AB is parallel to EH; also in the triangle IEH, since IE and IH are bisected, DC is parallel to HE, (prop. 7 of deduc.); ∴ DC is parallel to BA; for the same reason AC is parallel to BD; ∴ ABCD is a parallelogram.

PROP. 114. THEOR.

In figure sixty-nine, if the right lines (DG, EO, and LF) *be drawn, the triangles* (DAG, EBO, and LCF) *thus formed are equal to one another.*

Produce AB to K; then the angle CBK is evidently = to OBE; but CBK is the supplement of CBA; ∴ OBE is the supplement of CBA. Then in the triangles BOE and BCA, the sides OB and BE are respectively = to CB and BA, and the contained angles ABC and OBE supplemental; ∴ the triangles OBE and CBA are =, (prop. 44 of deduc.); for the same reason the triangles DAG and ABC are =, and ABC and FCL are =, (by prop. 4, 1. Elr.).

PROP. 115. PROB.

Given the excess of the perpendicular of an equilateral triangle above half the side, to construct it.

Fig. 70.

Upon the given excess AB describe an equilateral triangle ACB; draw CH at right angles to it; make HI = to HA, and IE = to IC; upon CE describe an equilateral triangle DEC, and produce CA to F; CF will be at right angles to DE, and AF will be = to half CE; join AI and FI. Then, because AH is = to IH, and AHI a right angle, each of the angles HAI and HIA is an angle of 45°; ∴ the angle IAF is an angle of 75°; and because the angle FCE is an angle of 30°, and DCE an angle of 60°, DCF is an angle of 30°; ∴ CF is at right angles to DE, and FE is half DE; ∴ FI is = to EF; ∴ FIE is an equilateral triangle; and, since FIE is an angle of 60°, and HIA of 45°, FIA is an angle of 75°, and is ∴ = to FAI; ∴ FA is = to FI, and ∴ to IE; ∴ CDE is the required triangle.

PROP. 116. THEOR.

If the hypotenuse (AB) *of a right angled triangle* (ABC) *be produced, and perpendiculars* (DM and OK) *be let fall on the produced line from the angles* (FDA and LOB) *of the squares described on the sides, those perpendiculars shall be respectively equal to their adjacent segments* (AP and BP) *of the hypotenuse, and the intercepts* (MA and KB) *between the perpendiculars and hypotenuse, are each equal to the perpendicular* (CP) *from the right angle on the hypotenuse.*

Fig. 69.

Because AL is parallel to BO, the angle OBK is = to LAK, and ∴ to BCP; then in the triangles BCP and OBK, the angles OBK and BKO of the one are respectively = to BCP and CPB of the other, and CB is = to BO; ∴ OK is = to BP, and KB to CP (prop. 26. 1. Elr.); for the same reason DM is = to AP, and MA to CP.

PROP. 117. THEOR.

In figure sixty-nine, the squares of the joining lines (DG, EO and LE) *are equal to six squares of the hypotenuse* (AB).

Produce DA and OB to meet perpendiculars in S and N.

Because CBN is a right angle, and also EBA, if NBA be taken away, the remainders EBN and CBA are =; ∴ the triangles BNE and BCA are similar and equal; for the same reason ASG and ACB are similar and =; ∴ ON is bisected in B, and also DS is bisected in A; ∴ (props. 43 and 47, 1. Elr.) the $\square^2$ of OE is = to four times the $\square^2$ of OB or BC + the square of NE or CA; for the same reason the $\square^2$ of DG is = to four times the $\square^2$ of DA or of AC + the $\square^2$ of SG or of CB; ∴ the $\square^2$rs of DG and OE together are = to five times the $\square^2$rs of AC and CB together, or to five times the $\square^2$ of AB; ∴ the squares of DG, OE and FL* together are = to six times the $\square^2$ of AB.

* FL is evidently = to AB (prop. 8. 1. Elr.).

PROP. 118. THEOR.

If two right angled triangles (ABC and ECD) *be similar, the square described upon the sum of the hypotenuses is equal to the squares described upon the sum of the bases, together with the squares described upon the sum of the other two sides.*

Fig. 71.

Place the given triangles so, that the bases BC and CD may be *in directum;* produce BA till the produced part AL is = to CE; join LE.

Since EC is = and parallel to LA, LE is = and parallel to AC; ∴ AE is a parallelogram; ∴ the angles LAC and LEC are =; also the angles BAC and CED are = (by hypoth.); but LAC and BAC are = to two right angles; ∴ LEC and DEC are = to two right angles; ∴ LE and ED are *in directum.*

Then, since LBD is a right angled triangle, the $\square^2$ of LD is = to the $\square^2$rs of LB and BD; but LD is = to the sum of the hypotenuses, BD the sum of the bases, and LB the sum of the sides; ∴ &c. &c.

PROP. 119. PROB.

In the greater side (BD) *of a scalene right angled triangle* (BDE) *to find a point from which, if two right lines be drawn, one to the opposite angle* (BED) *and the other to the hypotenuse* (BE), *their sum shall be a minimum.*

Fig. 72.

ANALYSIS.

Suppose the sum of EF and FG to be a *minimum;* produce ED, till DC is = to DE, join FC; it is evidently = to FE; then, since FC is = to FE, and the sum of FE and FG the least that can be drawn from any point in DB, the sum of CF and FG is the least that can be drawn from the point C to BE; ∴ CF and FG are *in directum* and at right angles to BE.

If then, you produce ED, till DC is = to it, draw CG at right angles to BE, join FE, the sum of FG and FE is less than the sum of any other two lines drawn from any other point in BD.

For, draw any other lines OL and OE, join OC and LC. Then CG is evidently = to FE and FG; for the same reason CO and OL are = to EO and OL; but CO and OL are greater than CL, and CL is greater than CG, because CGL is a right angle; ∴ CO and OL or EO and OL are much greater than CG or EF and FG; ∴ &c. &c.

PROP. 120. PROB.

Within a given quadrilateral figure (DECG) *to find a point from which the sum of the lines drawn to the angles shall be a minimum.*

Fig. 52.

Draw the diagonals CD and GE: their sum is less than the sum of any other lines drawn from any point O to the angles.

For DC is less than DO and OC together; ∴ SG, SC, SE, and SD, are together less than OD, OG, OC and OE together; ∴ the point S, in which the diagonals cut one another, is the point required.

PROP. 121. THEOR.

In figure sixty-nine, the right lines (DB and CG) *cut one another at right angles.*

ANALYSIS.

Suppose that B *s t*, one of the angles at the point of section, is a right angle, it is = to the angle *t* A G, and the angles B *t s* and G *t* A are = (prop. 15, 1 Elr.) ∴ the triangles G A *t* and B *s t* are similar ; ∴, if this is proved, and the proposition is true.

Then in the triangles BDA and GAC, the sides BA and AD are respectively = to GA and AC, and the angle GAC to BAD, ∴ the triangles are = in every way ; ∴ the angles AGC and ABD are =. Then in the triangles B*ts* and G A *t* the angles *s* B *t* and A G *t* are = (by demon.) and the angles A *t* G and B *ts* are = (by 15, 1 Elr.) ∴ the angles *t s* B and *t* A G are = (cor. 32, 1 Elr.) but *t* A G is a right angle, ∴ *t s* B is a right angle and ∴ the other angles at the point *s* are right angles.

PROP. 122. PROB.

To inscribe in a given square (DCG) *a right line parallel to a right line* (LG) *given in position and equal to* (FL) *given in magnitude, not greater than the diameter of the square.*

Fig. 11.

Draw from any angle BDE of the given $\square^2$ DK parallel to LG, make DO = to FL, through O draw OQ parallel to DE and through Q draw QP parallel to KD : it is = to FL and parallel to FG; the demonstration is evident.

PROP. 123. THEOR.

If the sides of a square (ADFC) *be bisected and right lines* (a D, C c, d F, A b) *be drawn from the points of bisection to the opposite angles* (FDA, CFD, DAC, ACF) *the figure* (e g k i) *formed by those drawn lines is a square, and is the fifth part of the entire figure.*

Fig. 73.

In the triangles *d* C F and C A *c* the angles A C *c* and C F *d* are evidently = to one another ; and the angle F *d* A is = to the angles *d* C F and C F *d*, and also to the angles *d* C *i* and C *i d*, ∴ the angle C *i d* is = to *d* C F, ∴ C *i d* is a right angle, ∴ *e i k* is a right angle ; for the same reason the angles *i k g*, *k g e*, *g e i* are right angles, ∴ the figure *e g k i* is rectangular. Then in the triangles F C *i* and DF *e* the sides D *e*, F *i* are evidently =, and since F C isbis ected in *a* and *a e* parallel to C *i*, F *i* is bisected in *e* ; for the same reason D *e* is bisected in *g*, ∴ *i e* is = to *eg* and ∴ to the opposite sides $i\ k$, $k\ g$, ∴ $i\ k\ g\ e$ is a square.

And each of the triangles $a\ e$ F, $b\ g$ D, $c\ k$ A, $d\ i$ C is the fourth part of each of the = triangles F C i, C A k, A D g, D F e, ∴ the four small ones are = and are together = to the fourth part of the figure outside $e\ i\ k\ g$: bisect $e\ i$ in o, then the triangle $e\ o\ g$ is evidently the fourth part of $e\ i\ k\ g$, and the triangles $g\ e\ o$ and F $e\ a$ are evidently = to one another, ∴ the four F $a\ e$, C $d\ i$, A $c\ k$, D $b\ g$, are together = to $e\ i\ k\ g$; ∴ it is the fourth part of the figure outside itself, and ∴ the fifth part of the $\square^2$ F C A D.

PROP. 124. THEOR.

In the theorem of Pappus, (vide Elrington, page 158,*) the right line* (B n) *drawn from one extremity of the side* (BA) *of the given triangle* (ABC) *equal and parallel to the right line* (CV) *joining the vertex and point of concourse of produced sides, shall fall on the produced side* (OV.)

Fig. 69.

For, if it do not, let it fall at X, draw XV.

Because BX is = and parallel to CV, the right line VX is parallel to FB, (prop. 33, 1 Elr.) but VO is also parallel to it, which is absurd, it is ∴ evident that the right line must fall on OV.

PROP. 125. THEOR.

In the theorem of Pappus, if the given triangle (ABC) *be right angled, the parallelogram* (n B A r) *described on the hypotenuse shall be a square, if those* (FA and LB) *described on the sides be squares, which shows that the* 47*th is a particular case of it.*

Fig. 69.

For, in the triangles BCA and CLV, the sides BC and CA of the one, are respectively = to CL and LV of the other, and the angle BCA is = to CLV; ∴ BA is = to CV, and ∴ to B n; ∴ the figure n B A r is equilateral; and, because the angle VCL is = to CBA, the angles LCV and BCT are together = to a right angle; ∴ LCV LCB and BCT are together = to two right angles; ∴

TC and CV are *in directum;* ∴ TC is parallel to B n; ∴ the angles n BC and BCT are = ; ∴ the angle n B A is a right angle; for the same reason the angle B A r is a right angle ; ∴ the figure B A r n is a square.

PROP. 126. THEOR.

Prove the three angles of any triangle (ABC) *equal to two right angles without producing a side.*

Fig. 1.

Draw through the vertex A, any right line FG parallel to the base BC.

Then the three angles BAF, BAC, and CAG are = to two right angles, (prop. 13. 1 cor. Elr.) ; and because FG is parallel to BC the angles FAB and ABC are =, also the angles GAC and ACB ; ∴ ABC,BCA and CAB are together = to two right angles.

PROP. 127. PROB.

In the ninth proposition of first book of Elrington, if the diagonals (AF *and* DE) *be equal, what will the measure of the given angle* (BAC) *be?*

Fig. 22, plate 1 Elr.

If the right line FA is = to DE, it is = to FE ; ∴ the angles FAE and FEA are = ; but the angle AFE is an angle of 30° ; ∴ each of the angles FAE and FEA is an angle of 75° ; for the same reason each of the angles FDA and FAD is an angle of 75° ; ∴ the given angle BAC is an angle of 150 degrees.

PROP. 128. PROB.

To construct an isosceles triangle, whose vertical angle shall be four times one of its base angles.

Fig. 57. ANALYSIS.

Suppose that MIL is the required triangle having the angle MIL = four times the angle IML.

Bisect the angle MIL by IP and produce it indefinitely ; draw to meet the production MO and LO making the angles IMO and ILO each = IML or ILM. The angle PIM is = to twice PMI, i. e. to PMO, and it is also =

to IOM + IMO; ∴ IOM and IMO (= ILO and IOL) are = to one another; ∴ the angles MOL, OLM and LMO are = to one another; ∴ OML is an equilateral triangle.

If therefore you construct an equilateral triangle OML, bisect the angles OML and OLM, the right lines MI and LI, bisecting them, form with ML the required triangle IML, having its vertical angle MIL four times one of its base angles.

The demonstration is evident.

PROP. 129. THEOR.

If two right angled triangles (ACG and FDE) *be similar, and if the right line* (CB) *drawn from the right angle* (ACG) *of one, perpendicular to the hypotenuse* (GA) *be equal to a side* (FD) *of the other, the hypotenuse* (FE) *of the latter, shall be equal to a side of the former.*

Fig. 55.

For, since the triangles are similar, the angles CAG and DFE are = ; also the angles DEF and CGA; and since CB is a perpendicular, the angle BCG is = to GAC; ∴ BCG is = to DFE; also the angles CBG and FDE are = being right angles, and the perpendicular CB is = to the side FD; ∴ (prop. 26. 1. Elr.) the triangles CBG and FDE are = in every respect; ∴ the side CG shall be = to the hypotenuse FE.

PROP. 130. PROB.

To draw from the vertex of a scalene triangle (BCD) *to the base a right line, which shall exceed the less side* (BC) *as much as it is exceeded by the greater* (CD).

Fig. 38.

ANALYSIS.

Suppose that CG is the line required to be drawn, and that it exceeds BC as much as it is exceeded by CD; make CF = to CG; CF exceeds CB by a part = to FD; make FE = to DF, then CE is = to CB; ∴ ED is the difference between CD and CB; ∴ EF is half the difference.

If therefore you make CE = to CB and bisect the difference ED in F, EF is evidently = to the line required to be drawn; ∴ from C as centre, and FS as radius, describe a circle; the point G, where it cuts the base, is the point to which the line is to be drawn.

PROP. 131. THEOR.

The right line (DF) *drawn from the vertex of a scalene triangle* (ADB) *to bisect the base, is greater than the right line* (DC) *bisecting the vertical angle and produced to the base.*

Fig. 17.

Since the angle DBA is greater than DAB, and DC bisects the vertical angle, CA is greater than CB, (prop. 19. dedu.); ∴ DF falls between DA and DC. Then the angle DCA is = to the sum of the angles CBD and BDC; ∴ to the sum of the angles CBD and CDA; and the angle BFD is = to the sum of the angles FDA and FAD; but FAD is less than FBD, and FDA less than CDA; ∴ BFD is less than FCD; ∴ DF is greater than DC.

PROP. 132. THEOR.

The two sides (AD and DB) *of any triangle* (ADB) *are together greater than double the right line* (DC) *bisecting the vertical angle and drawn to the base.*

Fig. 17.

If the given triangle is isosceles, the line bisecting the vertical angle also bisects the base; then it is evident that (prop. 94 of deduc.) the sum of the sides AD and DB is greater than double DC.

But, if the triangle be scalene, then the segment CA of the base adjacent to the greater side is greater than the segment CB, (prop. 19, deduc.); bisect AB in F; join F and D. Then FD is greater than CD, (prop. 130, deduc.); but the sides AD and DB are together greater than double of FD; ∴ they are much greater than the double of CD.

PROP. 133. PROB.

Given of any triangle, one side, an angle at it, and the sum of the other two sides, to construct it.

Fig. 31.

ANALYSIS.

Suppose that ABD is the required triangle: let AB be the given side, ABD the given angle, and BD = to the given sum; join D and A.

Then, since BD is = to the sum of BC and CA, CD is = to CA; ∴ the angles CAD and CDA are =.

If therefore you draw BA = to the given side, make the angle ABD = to the given angle, make BD = to the given sum, join DA and make the angle DAC = to CDA, it is evident, that ACB is the triangle required.

PROP. 134. THEOR.

To find a point in the base (AD) *of a scalene triangle* (ABD), *through which if a line be drawn at right angles to one of the sides* (BA), *the segment of the base between that side and the required point, shall be equal to the segment of the perpendicular between the point and the other side* (BD) *produced.*

Fig. 32.

ANALYSIS.

Suppose that I is the point required to be found; that CH is at right angles to AB, and that CI is = to IA; then, because CI and IA are =, the angles IAC and ICA are =; draw AX parallel to CH: because CH is parallel to XA, the angles CAX and ACI are =; ∴ CAX and CAD are =, and also the angle XAB is a right angle.

If therefore you draw AX at right angles to BA, bisect the angle DAX, draw the right line AC bisecting it to meet BD produced, and through C draw CH parallel to XA; the point I where it cuts AD is the required point.

For, since HC is parallel to AX, the angles HCA and CAX are =; ∴ ICA and IAC are =; ∴ IC is = to IA; and the angle CHA is evidently a right angle; ∴ &c. &c.

PROP. 135. THEOR.

If the base (BD) *of a right angled triangle* (BDG) *be divided into any number of equal parts* (DE, EF, and FB), *and right lines be drawn from the points of section to the opposite angle* (BGD), *the angle* (EGD) *subtended by the segment* (ED) *adjacent to the right angle, is greater than either of the remaining ones, and the segment nearest to this shall subtend a greater angle than the more remote.*

Fig. 23.

For, since the angle BDG is right, the angle DBG is less than it; ∴ (prop. 19. dedu.), if the angles EGD and EGF were =, EF would be greater than ED; *à fortiori*, if the angle EGF were greater than EGD, EF would be much greater than ED; but it is = to it; ∴ the angle FGE must be less than EGD; for the same reason the angle BGF is less than FGE.

PROP. 136. PROB.

In a given square (DC) *to inscribe an equilateral triangle, one of whose angular points shall be in an angular point* (B) *of the given square.*

Fig. 11.

Draw a diagonal BE; at the point B and with the right line BE make the angle EBP = to half the angle of an equilateral triangle; make the angle EBI = to EBP; join IP; IBP is the required triangle.

For, in the triangles DBP and IBC, the side DB is = to the side BC, and the angles BCI and CBI of the one are respectively = to BDP and PBD of the other; ∴ BP is = to BI; ∴ the angles BIP and BPI are =; but they are together = 120°; ∴ each of them is 60°; ∴ the triangle BPI is equilateral.

PROP. 137. PROB.

To find a point in the side (BD) *produced, of any triangle, from which a right line drawn at right angles to the other side* (BA) *and cutting the base* (AD), *may be equal to the base.*

Fig. 32.

ANALYSIS.

Suppose that HC is the line required: through A draw AO = and parallel to HC; through O draw OC; OC is = and parallel to AH.

If therefore through the extremity A of the side to which the line is to be drawn, you draw AO at right angles to it and = to the base; through O draw OC parallel to AB; C is the point required. For, draw CH parallel to AO; it is evidently = to it, and is ∴ = to the base AD; it is also at right angles to AB, since it is parallel to AO and OAB being a right angle.

PROP. 138. THEOR.

If any two points (*e* and F) *be taken in the opposite sides of a parallelogram* (CEHG) *at equal distances from the opposite angles* (ECG and EHG), *and if two other points* (*k* and *o*) *be taken in the other two sides at equal distances from the same angles, the right lines* (*eo*, *o*F, F*k*, and *ke*) *connecting those points form a parallelogram.*

Fig. 51.

For, in the triangles *k* C *e* and *o* HF, the sides *k* C and C *e* of the one are respectively = to *o* H and H F of the other, and the contained angles are =; ∴ *e k* is = to F *o*; and because C *e* is = to HF, *e* G is = to FE; for the same reason E *k* is = to *o* G, and the contained angles *o* G *e* and F E *k* =; ∴ *k* F is = to *e o*; ∴ *e o* F *k* is a parallelogram.

PROP. 139. THEOR.

If one of the diagonals (CB) *of a given parallelogram* (DBEC) *be equal to a side* (DB or CE) *of the parallelogram, the other diagonal* (DE) *shall be greater than any side of the figure.*

Fig. 15.

Because CB is = to CE, the angles CBE and CEB are = to one another; ∴ the angle CBE is greater than the angle DEB; much greater then is the angle DBE than DEB: for the same reason the angle DBE is much greater than the angle EDB; ∴ DE is greater than either of the sides DB or BE, and ∴ than either of the sides DC or CE.

PROP. 140. PROB.

To describe a circle through three given points (A, C, and G) *that are not in directum.*

Fig. 27.

ANALYSIS.

Suppose it done; connect the given points by the right lines GA, AC and CG, and also the centre of the circle I and the given points by IA, IG and IC, those lines are = to one another (def. 15 and 16); let fall from I on the sides AC, CG and GA, the perpendiculars IB, IE and

IF; then, in the triangles AIF and CIF, the side IA is = to IC, and the angles IAF and IFA of the one respectively = to ICF and IFC of the other; ∴ (prop. 26, 1 Elr.) AF is = to FC; for the same reason CE is = to EG, and GB to BA.

If therefore you bisect the lines connecting the given points, and draw perpendiculars from the points of bisection to meet; the point, in which they meet, is evidently the centre of the circle required.

PROP. 141. PROB.

Given the base (AC) *to find the locus of the vertices of an infinite number of triangles, the differences of the squares of whose sides shall be equal.*

Fig. 19. ANALYSIS.

Suppose ABC and AEC to be any number of triangles, the difference of the squares of whose sides are =; then (by cor. 4, prop. 47, 1 Elr.) the difference between the squares of AB and BC is = to the difference between the $\square^2$rs of the segments AD and DC of the base, made by the perpendicular BD from the vertex on the base; also the difference between the sides AE and EC is = to the difference between the $\square^2$rs of the segments of the base; but, since the differences are =, the segments must be =; ∴ the locus of the vertices lies in a line drawn at right angles to the base.

PROP. 142. PROB.

Given any triangle (BAI), *one of whose sides* (AI) *is produced, to let fall from the produced side, on the other* (AB), *a perpendicular, which shall be bisected by the base* (BI).

Fig. 67. ANALYSIS.

Suppose that GF is the line required; draw through B the right line BD parallel to GF and to meet the produced side; draw AH and produce it to E; since GF is bisected by AE, BD is also bisected by it (prop. 14, deduc.).

Therefore let BD be drawn to meet AF produced and at right angles to AB; bisect BD in E; join EA; it is evident the right line FG, drawn through H parallel to BD, is at right angles to AB and bisected by BI.

PROP. 143. THEOR.

The right line (DA), *drawn from the point of concourse* (D), *of two lines* (DB and DC) *bisecting any two angles of a triangle* (ABC), *to the remaining angle* (BAC), *bisects it also.*

Fig. 50.

Draw DE, DF and DG at right angles to the sides; they are evidently = to one another; and because in the triangles AED and AGD the side AD is common, the angles AED and AGD right angles, and the angles GAD and EAD of the same affection, the triangles are = in every respect (prop. 77, dedu.); ∴ the angles GAD and EAD are =; ∴ &c.

PROP. 144. THEOR.

That point within a triangle (MOL), *from which lines* (IO, IL and IM), *drawn to the opposite angles, trisect the triangle, is the occurse of three lines bisecting the sides.*

Fig. 57.

Because ML is bisected in P, the triangle MPO is half of MLO; for the same reason the triangle DML is half MLO; ∴ DML and MPO are =; if then the figure MPID be taken from both the triangles, PIL and DIO will remain = to one another; and, because ML is bisected in P, the triangles PIM and PIL are =; ∴ PIM and DIO are =; for the same reason DIM and PIL are =; ∴ the triangle MIO is = to MIL; in like manner the triangle LIO can be proved = to LIM; ∴ the three triangles LIM, MIO and OIL are =; ∴ the point of occurse of those lines is the point, from which the lines drawn to the angles trisect the triangle.

PROP. 145. PROB.

Given of any triangle the base, the sum of the sides, and the vertical angle, to construct it.

Fig. 28.

ANALYSIS.

Suppose CAB to be the required triangle, CB the given base, CK the given sum, and CAB the given angle, draw KB; it is evident, that AK is = to AB, and ∴ the angles

AKB and ABK are = ; but the angle CAB is = to AKB and ABK ; ∴ AKB is = to half the given angle.

Therefore, at one extremity K of the given sum, make the angle CKB = to half the vertical angle, and from the other extremity C inflect the right line CB = to the given base ; make the angle KBA = to AKB, it is evident that CAB is the required triangle.

PROP. 146. THEOR.

If the sides (EB and GA) *of a square, described on the hypotenuse of a right angled triangle, be produced to meet the sides* (DF and OL) *or those produced, if necessary, the triangles* (B O *n* and *r* D A), *cut off by them, are equal and equiangular to the given triangle.*

Fig. 69.

Because the angle *n* B A is a right angle, and also the angle OBC, ∴ if the angle *n* B C be taken away, the angles O B *n* and C B A remain = to one another ; ∴ in the triangles O B *n* and C B A, the angles O B *n* and B O *n* of the one are respectively = to CBA and BCA of the other, and the side OB is = to BC ; ∴ the triangles O B *n* and C B A are = and equiangular ; in like manner the triangles D *r* A and C B A can be proved equiangular and =.

PROP. 147. THEOR.

The right line (BD) *joining the acute angle* (CBA) *of a right angled triangle, with the opposite angle* (FDA) *of a square, described upon the side subtending this acute angle, will cut from the triangle an obtuse angled triangle* (B *e* A), *which shall be equal to that* (*e* C F) *cut off from the square, by a line* (*e* F) *drawn from the intersection of the connecting line with the side to the angle* (BFD) *opposite said connecting line.*

Fig. 69.

Draw the diagonal CD.

Then, since the triangles C *e* D and C *e* F are on the same base C *e* and between the same parallels, they are = ; and also, because the triangles DAB and DAC are on the same base DA, and between the same parallels AD and FB, they are = ; if ∴ the common part A *e* D be taken away, the remaining triangles D C e and A e B shall be $=$; $\therefore$ A e B is $=$ to e F C; and the angle A e B is evidently obtuse, since it is greater than e C B, which is a right angle.

PROP. 148. PROB.

Given the lengths of the three lines, which, drawn from the angles, bisect the opposite sides, to construct the triangle.

Fig. 57.

Construct a triangle OML, whose sides shall be double the given bisectors; bisect MO in D, join DL, make LI two-thirds of LD; then ID is one-half LI; produce ID until DB is $=$ to it; join IO and BO; BOI is the required triangle.

For DO is evidently $=$ to one of the given bisectors, since it is one-half MO, which is double one of them; draw I e parallel to LO; then, in the triangle BLO, the side BL is bisected in I, and I e drawn par. to LO; the side OB is also bisected in e, and I e is $=$ to half LO, (prop. 6. dedu.); $\therefore$ I e is $=$ to another of the bisectors; $\therefore$ it is evident that the right line B g, drawn through their point of concourse, bisects IO, and must be $=$ to the third bisector.

PROP. 149. PROB.

Given any quadrilateral figure (GKML), *it is required to bisect it by a right line drawn from a given point* (A) *in one of its sides.*

Fig. 54.

Connect the given point A with the opposite angle; bisect the figure AKML by A e; upon AE and at the side of it, at which the greater part of the given figure lies, describe a triangle A e M $=$ to AKG; bisect this triangle by the right line A o; this right line A o evidently bisects the given figure.

PROP. 150. THEOR.

The diagonals (OC and CD) *of the squares described on the sides of a right angled triangle, which terminate in the vertex of the triangle, lie in directum.*

Fig. 69.

Because, in the triangle LOC, the side LO is = to LC, and the angle OLC a right angle, the angle LCO is half a right angle; for the same reason the angle FCD is half a right angle, and the angle LCF is a right angle; ∴ the three LCO, LCF, and FCD are = to two right angles; ∴ (prop. 14, 1 Elr.) the right lines OC and CD are *in directum.*

PROP. 151. THEOR.

In figure sixty-nine, the diagonal (CV) *of the parallelogram* (LCFV), *formed by producing the sides* (DF and LV) *of the squares described on the sides of a right angled triangle, is in directum with the perpendicular* (CT) *let fall from the vertical angle on the hypotenuse.*

Because, in the triangles CLV, and BCA, the side CL is = to BC, the side LV (= CF) = CA, and the angle CLV = BCA, the angle ABC shall be = to VCL; but the angle ABC with BCT is = to a right angle; ∴ VCL and BCT are = to a right angle; but LCB is a right angle; ∴ the three TCB, BCL, and LCV are = to two right angles; ∴ TC and CV are *in directum.*

PROP. 152. THEOR.

If right lines (DP, KG, and LI), *drawn at right angles to the sides of an equilateral triangle* (ABC), *form a triangle* (POF), *it also shall be equilateral.*

Fig. 74.

Because ILC is a right angle, it is = to the sum of the angles LCI and CIL; but LCI is an angle of 60 degrees; ∴ CIL is an angle of 30°; and since ODI is a right angle, and DIO an angle of 30°, IOD is an angle of 60°; ∴ its vertically opposite angle POF is an angle of 60°; for the same reason OFP is an angle of 60°; ∴ POF is an equilateral triangle.

PROP. 153. THEOR.

If right lines (RT, IH, and US) *be drawn from the sides of any triangle* (OPQ) *with the same inclination, i. e. making the angles* (TRP, HIQ, and SUO) *equal to one another, the triangle* (e d o) *formed by them shall be equiangular to the whole triangle.*

Fig. 75.

For the angle TRP is = to the angles RTQ + RQT; but HIQ is = to TRP, ∴ HIQ is = to RTQ + RQT; and also to I T o + I o T; ∴ IOT is = to RQT; ∴ its vertically opposite angle d o e is = to PQO; in like manner e d o can be proved = to OPQ; ∴ o e d is = to POQ; ∴ &c. &c.

PROP. 154. THEOR.

In figure sixty-eight, the parallelogram (ABCD) *is equal to half the difference of the triangles* (EGF and HGI.)

For, draw EH and FI and bisect them in *b* and *d*, join *b*, B and *d*, C and also A *b*, *b* D, D *d*, and *d* A.

Then it is evident, that the triangles A C *d*, and B *b* D are = ; ∴ they are together = to half the triangle IEF; for the same reason the triangles C D *d*, and *b* B A are together = to half the triangle HEF; and the parallelogram A *b* D *d* is = to half the figure EHIF. Then EHIF is = to HEG + IGF + EGF + HGI, which are = to twice A *b* D *d*, which is = to HEG + EGF + IGF + EGF + twice ABDC, since the four small triangles D B *b*, *b* B A, A *d* C, *d* C D are together = to half the sum of the triangles HEF, IEF; ∴ HEF + IEF + twice ABDC is = to twice A *b* D *d* or to EHIF; but IGH + HGE + EGF + FGI is also = to EHIF; ∴ IGH and EGF are = to twice EGF + twice ABDC ∴ HGI − EGF is = to twice ABDC; ∴ ABDC is = to half the difference of the triangles HGI, and EGF.

END OF FIRST BOOK.

THE

SECOND BOOK,

IN

GENERAL TERMS.

PROPOSITION 1. THEOR.

If there be two right lines, one of which is divided into any number of parts, the rectangle under the two lines is equal to the sum of the rectangles under the undivided line and the several parts of the divided line.

Draw from either extremity of the divided line, a line at right angles to it, and = to the undivided line, and complete the rectangle; then, through each point of bisection of the divided line, draw a line parallel to the perpendicular, to meet the opposite side.

Then it is evident, that those parallel lines are = to the perpendicular, and ∴ to the undivided line; also that the whole rectangle is = to a rect. under the given lines, and is divided into as many rectangles as there are segments in the divided line, and ∴ into rectangles under the undivided line and the several parts of the divided line.

Cor. † Hence and from prop. 34, b. 1, it appears, that the area of a rectangle is found by multiplying its altitude into its base; ‡ and from prop. 35 and 36, b. 1, it also appears, that the area of any parallelogram is found by multiplying its altitude into its base; § and from prop. 37, b. 1, that the area of a triangle is found by multiplying its altitude into half its base.

Those propositions may be reduced to numbers, thus: suppose the divided line 16 feet is divided into 8, 6, and 2 feet; then 16 multiplied by the undivided line 6 feet, is = to the sum of the products of 8×6, 6×6 and 2×6.

For $16 \times 6 = 96$, and $8 \times 6 = 48$, $+ 6 \times 6 = 84$, $+ 2 \times 6 = 96$.

From this it is evident, that the product of any two numbers is = to the sum of the products of either of them multiplied into the several parts, into which the other may be divided; for, if 246 is to be multiplied by 6, the product will be the same as the sum of the products of 200×6, 40×6 and 6×6; from this principle, the arithmetical process of multiplication is performed, which is nothing more than common addition; for 246 multiplied by 6 is the same as 246 placed 6 times under one another and the entire added together.

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† For if the altitude be 8 feet and the base 10, it will be = to 10 rectangles, each of whose bases is one foot and altitude eight; the area of each of these would be 8 square feet; ∴ the area of the whole rect. will be 80 square feet. From this it is evident, that the rect. under any line and its tenth part is = to ten times the $\square^2$ of that part.

‡ For it is = to the area of a rectangle on the same base and of the same altitude, i. e. between the same parallels.

§ For it is = to half the area of a parallelogram on the same base and of the same altitude.

PROP. 2. THEOR.

If a right line be divided into any two parts, the square of the whole line is equal to the sum of the rectangles under the whole line and each of the parts.*

Describe on the given line a $\square^2$ (prop. 46, 1.), and through the point of section draw a line parallel to one of the sides.

Then it is evident (prop. 1, b. 2.), that this $\square^2$ is = to the rectangles into which the parallel line divides it, which are = to the rectangles under the whole line and the parts of it.

Otherwise thus:

Assume a right line = to the given line.

Then the rectangle under the assumed line and given line, or the $\square^2$ of the given line, is = to the sum of the rectangles under the assumed line and each of the parts, into which the given line is divided (prop. 1, 2).

* The line may be divided into any number of parts, and the proposition is equally true.

In numbers, suppose the given line is 18, and that it is divided into 12 and 6 feet; then the $\square^2$ of 18, viz. 324, is = to $12 \times 18 = 216, + 6 \times 18 = 324$; or if the right line be divided into any number of parts, viz. 5, 7, 4, 2; then the $\square^2$ of 18 is = to $18 \times 5 + 18 \times 7 + 18 \times 4 + 18 \times 2$. From this prop. it is evident, that the rectangle under any two lines is = to the $\square^2$ of the greater − the rectangle under the greater and difference. For, the $\square^2$ of the greater is = to the rect. under the lines + the rect. under greater and difference.

Thus in numbers 12×8 is = to $12 \times 12 - 12 \times 4$; i. e. $144 - 48 = 96$.

PROP. 3. THEOR.

If a right line be divided into any two parts, the rectangle under the whole line and either part, is equal to the square of that part and the rectangle under the parts.

On either part describe a square, and complete the rectangle under the whole line and this part.

This rectangle is = to the square of one part with the rectangle under the parts. For it is the rectangle under the whole line and that part, on which the $\square^2$ is described; if then the $\square^2$ be taken away, the remainder will be the rectangle under the parts; ∴ &c. &c.

Otherwise thus:

Assume a line = to either part.

Then, the rectangle under this assumed line and the whole line is = to the sum of the rectangles under the assumed line and each of the parts (prop. 1, b. 2.); but the rect. under the assumed line and part = to it, is the

$\square^2$ of that part; and the rect. under assumed line and other part is = to the rect. under the parts; ∴, &c. &c.

From this prop. we may infer, that the rect. under any two lines is = to the $\square^2$ of the less + the rect. under the less and difference.

Thus, in numbers 6×4 is = to $4^2 + 4 \times 2$.

From this and the former prop. may be inferred, that the $\square^2$ of the lesser of any two lines + the rect. under it and difference is = to the $\square^2$ of the greater − the rect. under it and difference.

Thus, in numbers, suppose 8 and 6 are the given quantities: then $6 \times 6 + 6 \times 2$ is = to $8 \times 8 - 8 \times 2$.

PROP. 4. THEOR.

If a right line be divided into any two parts, the square of the whole line is equal to the sum of the squares of the parts, and twice the rectangle under the parts.

On the given line describe a $\square^2$; draw a diagonal; draw through the point of section a right line parallel to one side of the $\square^2$ to meet the opposite side, and through the point, in which this parallel line cuts the diagonal, draw a line par. to the given line.

The entire $\square^2$ is divided into the squares about the diagonal, which are the $\square^2$rs of the parts, and the complements, each of which is a rectangle under the parts, &c. &c.

Otherwise thus:

The $\square^2$ of the whole line is = to the sum of the rectangles under the whole line and each of the parts (prop. 2, b. 2.); and each of those rectangles is = to a $\square^2$ of its respective part, and a rect. under the parts; ∴ the $\square^2$ of the whole line, &c. &c.

Cor. Hence it is evident, that the square of half the line is the fourth part of the square of the whole line; for, when the line is bisected, the rectangle under the parts is equal to the square of half the line.

NOTE.—This prop. may be thus reduced to numbers: suppose the given line 10, and the parts 6 and 4 feet; then 10×10 (= 100) is = to $6 \times 6 + 4 \times 4 + 6 \times 4 + 4 \times 6$.

From this prop. it is evident, if two right lines be given, that the $\square^2$ of their sum exceeds the sum of their $\square^2$rs by two rectangles under them; and if the lines be =, the $\square^2$ of their sum is double the sum of their $\square^2$rs.

PROP. 5. THEOR.

If a right line be cut into equal and also into unequal parts, the rectangle under the unequal parts, with the square of the intermediate part, is equal to the square of half the line.

Describe a square on the half line, which contains the intermediate part; draw a diagonal; through the point of unequal section draw a line parallel to a side of the square, and through the point in which it cuts the diagonal draw a line par. to the given line, to meet a line drawn through the other extremity of the given line, par. to a side of the $\square^2$.

Then, since the rectangles under the halves and lesser segment are =, and also the complemental rectangles =; the rectangle under the unequal segments is = to the gnomon formed by taking the $\square^2$ of the intermediate part from the $\square^2$ of half the given line; ∴ if the $\square^2$ of the intermediate part be added to each, they will be = to one another.

Otherwise thus:

The rectangle under the unequal segments is = to a rect. under half and lesser seg. with a rect. under intermediate part and lesser segment; but the rect. under half and lesser seg. is = to a rect. under intermediate part and lesser seg. together with the $\square^2$ of lesser seg.; ∴ two rectangles under intermediate and lesser seg. with the $\square^2$ of lesser are together = to a rect. under the unequal segments; if ∴ the square of intermediate part be added to both, two rectangles under intermediate and lesser segs. with the $\square^2$rs of inter. and lesser (viz. $\square^2$ of half the given line) are = to a rectangle under the unequal parts with $\square^2$ of intermediate.

Or thus in numbers:

Suppose the given line to be 12 feet, let it be divided equally into 6 and 6, and unequally into 8 and 4 feet, the intermediate part will be 2 feet.

Then $8 \times 4 + 2^2$ (= 36), is = to $6^2 = 36$.

Cor. 1. Hence it is evident, that, if a right line be bisected, the rectangle under the parts is greater than if it be cut unequally, and $\therefore$ the sum of the squares of the parts is less.

For then the rect. is $=$ to the $\square^2$ of half the line, and is $\therefore$ greater than if the line were cut unequally.

NOTE.—Since the lines are cut equally, the rectangles are a *maximum;* but they with the $\square^2$rs of the parts are $=$ to the $\square^2$ of the sum (prop. 4, 2.); and, if the line were cut unequally, the rect. would be less than those; but they with the $\square^2$rs of the unequal parts are also $=$ to the $\square^2$ of the sum; $\therefore$ the sum of the $\square^2$rs of the $=$ parts is less than that of the unequal parts. In numbers, 6 and 6 are $=$ parts; 8 and 4 unequal; then $6^2 + 6^2 = 72$ and $8^2 + 4^2 = 80$.

Cor. 2. If two equal right lines be so divided, that the rectangle under the segments of one be equal to the rectangle under the segments of the other, their segments shall be equal.

If one of the lines be bisected, the other also is bisected; for the rectangle under the parts is then equal to the $\square^2$ of the half line; whence it is evident, that the segments themselves are equal.

If they are not bisected, let them be cut unequally; bisect them; then, since the lines are $=$, their halves are $=$, and $\therefore$ the $\square^2$rs of their halves are $=$; but the rectangles under the unequal parts are also $=$; take them from the $\square^2$rs of the halves, and the remainders, viz. the $\square^2$rs of the intermediate parts, are $=$; $\therefore$ the intermediate parts themselves are $=$; $\therefore$ the sums of them and the halves are $=$; and also the differences between them and the halves are $=$; but these sums and differences are the segments.

Cor. 3. The rectangle under the sum and difference of two right lines is $=$ to the difference between their squares.

Because the rectangle of sum and dif. (of unequal parts) together with the $\square^2$ of the less (the interm.) is $=$ to the $\square^2$ of the greater (the half,) as is evident from the preceding proposition; for the half line is the greater, the intermediate part the less, and the lesser of the unequal segments the difference.

This might be proved (by prop. 3, 2.) since the $\square^2$ of the less $+$ the rect. under it and dif. is $=$ to the $\square^2$ of greater $-$ the rect. under it and dif.; $\therefore$ the difference between those $\square^2$rs is $=$ to the rect. under the greater and dif. + the rect. under the lesser and dif.; i. e. by 1, 2, the rect. of sum and dif.

Schol. 1. Hence it is also evident, that in a right angled triangle, the rectangle under the sum and difference of the hypotenuse and one side, is = to the square of the other side.

Schol. 2. Hence it is also evident, that if from any angle of a triangle, a perpendicular be drawn to the opposite side, the rectangle under the sum and difference of the sides is = to the rectangle under the sum and difference of the segments of the side, to which the perpendicular is drawn.

For the rectangle under the sum and difference of the sides is = to the difference between the $\square^2$rs of the sides; and the rect. under the sum and difference of the segments of the base is = to the difference between the squares of those segments, and these differences are =, (cor. 4, prop. 47, 1,); ∴, &c. &c.

Cor. 4. The difference between the squares of two sides of any triangle is = to double the rectangle under the remaining side and its distance between the middle point and the perpendicular, which is drawn to it from the opposite side.

For the difference between the $\square^2$rs of the sides is = to the difference between the $\square^2$rs of the segments of the side, upon which the perpendicular falls (cor. 4, prop. 47, 1,), and ∴ to the rectangle under the sum of the segments and the difference between them, which difference (when the perpendicular falls within the triangle) is double the seg. between the middle point and perpendicular; ∴ to double the rectangle under the sum and distance from middle point to perpendicular.

But, if the perpendicular falls outside the triangle, the segments are the sum of the side on which perpendicular falls and intercept between perpendicular and side, and said intercept; ∴ the difference is the side opposite the angle, from which the perpendicular is let fall.

Schol. Hence, given in numbers the sides of any triangle, we can find its area; divide the difference between the squares of two sides of the triangle by the remaining side; to half this side add half the quotient, and subtract the square of this sum from the square of the greater side; the remainder is the square of the perpendicular; thence the perpendicular itself is found, which multiplied into half the side on which it falls, gives the area of the triangle.

NOTE 1.—In the foregoing prop. the intercept is the segment between the points of equal and unequal section; and is $=$ to the difference between half the line and either of the unequal parts, and is $\therefore$ half the difference between the unequal parts.

2. It is evident from this proposition, that the excess of the $\square^2$ of half the sum of two lines above half their difference is $=$ to a rectangle under them; for half the given line is half the sum of the unequal parts, and the intermediate part is half their difference.

Hence and from prop. 7, we might infer, that the $\square^2$ of the difference of any two lines is $=$ to the excess of the sum of their squares above the double rectangle under them.

For the sum of the $\square^2$rs is $=$ to the $\square^2$ of the difference, two rectangles under the lesser and dif. with two $\square$ rs of the lesser; but two rectangles under less and dif. with two $\square^2$rs of lesser is $=$ to two rectangles under the given lines; $\therefore$ &c. &c.

NOTE 3. COR. 2. The more unequally a line is cut, the greater will be the intermediate part, and $\therefore$ the less will be the rectangle under the parts, the limits of which will be 0, and the $\square^2$ of half the line; the greatest limit of the sum of the $\square^2$rs is the $\square^2$ of the whole line, and the least is twice the $\square^2$ of half.

NOTE 4. By being given the rectangle under any two lines and their difference, we can find the lines.

For if the square of half the difference be added to the given rect., the sum will be $=$ to the $\square^2$ of half the sum, (for half the difference is $=$ to the intermediate part); then add to a side of this square (or its root) i. e. the half sum itself, half the difference, and you will have the greater line, and subtract the half dif. from said half sum, and you will have the lesser line.

In numbers, suppose the rect. to be 40 square feet, and the dif. to be six feet. Then the $\square^2$ of $3 = 9, + 40 = 49$, the square root of which is 7, which is half the sum of the given lines; then $7 + 3 = 10$, the greater line, and $7 - 3 = 4$, the lesser line.

5. Given the difference between any two lines, and the difference of their squares, to find the lines.

Describe on the difference of the lines a rect. $=$ to the dif. of the squares, its other side shall be $=$ to the sum of the required lines; for the difference of the $\square^2$rs is $=$ to a rectangle under their sum and difference. Or, divide dif. of $\square^2$rs by dif. of lines, and quotient is sum of lines.—(Lardner).

Thus in numbers, suppose the dif. of the lines is 8 feet, and the dif. of the $\square^2$rs 128 square feet; then 8 being one side of the rect. of 128 square feet, 16 must be the other side; the half of $16 +$ the half of $8 = 12$ the greater line; and the half of $16 -$ the half of $8 = 4$, the lesser line.

The difference of the $\square^2$rs of any two lines is greater than the $\square^2$ of their difference, by two rectangles under the less and difference; and is greatest, when the less is half the difference or one-third of the greater (the sum being supposed given.)

6. To cut a given right line so that the $\square^2$ of one part shall be $=$ to half the $\square^2$ of the given line. Describe on it an isosceles right angled triangle, and cut from the given line a part $=$ to a side of this triangle: the $\square^2$ of this part is evidently $=$ to half the $\square^2$ of the given line.

7. To cut a right line so that the rectangle under the segments shall be equal to the square of the intermediate part.

Bisect the given right line, and then cut its half (according to the foregoing note) so that the part, whose $\square^2$ is $=$ to half the $\square^2$ of the half line, be the intermediate part. For the $\square^2$ of the intermediate part, being $=$ to half the $\square^2$ of half the line, is $=$ to the rectangle under the unequal segments; ∴, &c.

8. To cut a line so that the rectangle under the parts shall be any multiple of the square of the intermediate part.

The rectangle, with the $\square^2$ of the intercept, is $=$ to the $\square^2$ of half the line; then suppose we want to cut it so that the rectangle may be three times the $\square^2$ of the intercept; let the rectangle be r, the $\square^2$ of the intercept s, and the $\square^2$ of half the line x.

Then $r + s = x$; $\therefore\ 3s + s\,(= 4s) = x$; $\therefore\ s = \frac{x}{4}$

i. e. The $\square^2$ of the intercept must be the fourth part of the $\square^2$ of half the given line; ∴ bisect half the line, and you have the required intermediate.

9. The sum of the squares of any two lines exceeds the difference of the squares by two squares of the lesser.

For the difference of the $\square^2$rs is = to a rect. under sum and dif. or to a $\square^2$ of the dif. and two rect. under lesser and difference; but those, together with two $\square^2$rs of the lesser, are = to the sum of the $\square^2$rs; ∴, &c. &c.

10. The sum of the squares exceeds the square of the difference by two rectangles under the given lines.

For the sum of the $\square^2$rs is = to a $\square^2$ of the difference, two $\square^2$rs of the lesser and two rect. under lesser and dif.; but two $\square^2$rs of the lesser, with two rectangles under lesser and dif., are = to two rect. under the lines.

11. The square of the sum exceeds the difference of the squares by two rectangles under the sum and lesser.

12. The square of the sum exceeds the square of the difference by four rectangles under the given lines.

13. The difference of the squares exceeds the square of the difference by two rectangles under the lesser and difference.

14. The square of the greater of two lines exceeds the rectangle under the greater and less, by a rectangle under greater and difference. Hence it is evident, that the rectangle under the lines exceeds the square of the less by the rectangle under the lesser and difference.

15. If a right line be cut twice unequally, the greater rectangle exceeds the less by a square of the greater intermediate part, minus a square of the lesser intermediate, or by two rectangles under lesser inter. and dif. of inter. together with a square of said dif.

16. The rect. under the segments of the hypotenuse (of a right angled triangle) made by a perpendicular, is = to a square of the perpendicular.

For the sum of the $\square^2$rs of the sides is = to the $\square^2$rs of the segments with two $\square^2$rs of the perpendicular, and also to the $\square^2$rs of the segments with two rectangles under the segments; ∴ those rect. must be = to the two $\square^2$rs of the perpendicular; ∴, &c. &c.

Hence the $\square^2$ of one segment + the rectangle under the segments, is = to the $\square^2$ of the conterminous side.

PROP. 6. THEOR.

If a line be bisected and produced to any point, the rectangle under whole produced line and produced part with the square of half the line, are equal to the square of the line made up of half and produced part.

Describe a $\square^2$ on the line made up of the half and part produced; from the common extremity of the given line and produced part draw a line par. to a side of the $\square^2$; from the other extremity of the produced part draw a diagonal; and, through the point in which it cuts the parallel line, draw a line $=$ and parallel to the whole produced line, and connect their other extremities.

Then, because the rectangles under the halves and produced parts are $=$, and also the complemental rectangles, the gnomon formed by taking away the $\square^2$ of half the given line is $=$ to the rectangle under the whole produced line and produced part; therefore if you add to both the $\square^2$ of half the given line, the rectangle under the whole produced line and produced part, together with the $\square^2$ of half the given line, is $=$ to the $\square^2$ of the line made up of the half and produced part.

Otherwise thus:

The rectangle under the whole produced line and produced part is $=$ to the sum of two rectangles under half the given line (which may be considered an intermediate part) and produced part, together with the $\square^2$ of the produced part; if ∴ you add the $\square^2$ of half (or intermediate) to both, the rectangle under the whole produced line and produced part, together with the $\square^2$ of half the given line, are $=$ to two rectangles under the half and produced part, together with the $\square^2$rs of half and produced part; but these latter together make up the $\square^2$ of the line made up of half and produced part; ∴, &c. &c.

In numbers, suppose the given line 8, and the produced part 4 feet, then the whole line is 12 feet: then $12 \times 4 + 4^2$ ($= 64$) is $=$ to 8^2 (the line made up of one half and produced part.)

Cor. If a right line be drawn from the vertex of an isosceles triangle to the base, the rectangle under the segments of the base, is $=$ to the difference between the square of this line and the square of either side.

Bisect the base and connect the point of bisection with the vertical angle.

Then the rect. under the former segments is $=$ to the difference between the $\square^2$ of half the base and the $\square^2$ of the intercept between the point of bisection and other point of section (prop. 5 and 6, b. 2.); but the dif. between the $\square^2$rs of half and intercept is $=$ to the dif. between the $\square^2$rs of the first drawn line and a side of triangle, ∴, &c.

But if the line drawn to the base be perpendicular to it, it bisects the base, and the rectangle under the segments of the base is the $\square^2$ of the half, and ∴ together with the $\square^2$ of the line drawn to base is = to the square of the side.

PROP. 7. THEOR.

If a right line be divided into any two parts, the squares of the whole line and either segment are equal to double the rectangle under the whole line and that segment, together with the square of the other segment.

Describe on the given line a $\square^2$, draw a diagonal, through the point of section draw a right line parallel to one side, and through the point in which it meets the diagonal draw a right line par. to given line.

Then the $\square^2$ of given line is = to a rect. under the whole line and one segment, a complemental rect. and a $\square^2$ of the other segment; add to both the $\square^2$ of the former segment: then the $\square^2$rs of given line and one seg. are together = to a rect. under the whole line and one seg. a complemental rect. and the $\square^2$rs of the segments; but the complemental rect. and added $\square^2$ is = to a rect. under the whole line and that segment; ∴ the $\square^2$rs of the whole line, &c. &c.

Otherwise thus:

The $\square^2$ of the whole line is = to the $\square^2$rs of the segments and two rectangles under the segments; add to both the $\square^2$ of one segment; then the $\square^2$rs of the whole line and one segment are together = to two rectangles under the segments, a square of one seg. and two $\square^2$rs of the other segment; but two rectangles under the segments with two $\square^2$rs of the segments are = to two rectangles under the whole line and that segment; ∴ &c. &c.

Thus in numbers: suppose the given line to be 12 feet, and that it is divided into 8 and 4 feet, then $12^2 + 4^2$ is = to twice $12 \times 4 + 8^2$.

For 12^2 (= 144) + 4^2 = 160, and 12×4 (= 48) + 12×4 (= 96) + 8^2 = 160.

PROP. 8. THEOR.

If a right line be divided into any two parts, the square of the sum of the whole line and either segment, is equal to four times the rectangle under the whole line and that segment together with the square of the other segment.

Produce the given line till the produced part is = to one segment; on the whole compound line describe a $\square^2$; through the point of section, and common extremity of given line and produced part, draw lines parallel to a side of the $\square^2$; through the other extremity of produced part draw a diagonal, and through the points, in which it meets the drawn lines, draw two other lines par. to the given line.

Then the $\square^2$ of the whole produced line is = to the $\square^2$ of one seg. of given line, four $\square^2$rs of the other and four rectangles under the segments; but four $\square^2$rs of one seg. with four rectangles under the segments, are together = to four rectangles under the given line and that seg. (of which are the $\square^2$rs) ; ∴ &c. &c.

Otherwise thus:

Produce the given line till the produced part is = to the adjacent segment: then the $\square^2$ of the whole produced line is = to a $\square^2$ of the given line, a $\square^2$ of the produced part (which is = to a $\square^2$ of one seg.) and two rectangles under the given line and produced part; the $\square^2$ of the given line is = to the $\square^2$rs of the segments and two rectangles under the segments; but one of those $\square^2$rs, the $\square^2$ of produced part, and two rectangles under the segments are together = to two rectangles under the given line and produced part; ∴ the $\square^2$ of the whole produced line is = to four rectangles under the given line and one segment, together with the $\square^2$ of the other segment.

Thus in numbers; suppose that the given line is 8 feet and the segments 6 and 2 feet; then the produced part will be 2 feet, and the whole produced line will be 10 feet. Then 10^2 (= 100) is = to $8 \times 2 + 8 \times 2 + 8 \times 2 + 8 \times 2 + 6^2 = 100$.

PROP. 9. THEOR.

If a right line be cut into equal and also into unequal parts, the sum of the squares of the unequal parts is equal to double the sum of the squares of the half and of the intermediate part.

The $\square^2$ of the greater segment is = to a $\square^2$ of half, a $\square^2$ of intermediate part and two rectangles under half and intermediate; add to both the $\square^2$ of the lesser segment; then the $\square^2$rs of the unequal segments are together = to the $\square^2$rs of half, of intermediate and of lesser seg. with two rectangles under half and intermediate; but two rectangles under half and intermediate with the $\square^2$ of lesser segment are = to a $\square^2$ of half and a $\square^2$ of intermediate, (prop. 7, 2); ∴ the sum of the $\square^2$rs of the unequal parts are = to two $\square^2$rs of half and two $\square^2$rs of intermediate part.

PROP. 10. THEOR.

If a right line be bisected and produced to any point, the square of the whole line thus produced, together with the square of the produced part, is equal to double the square of the line made up of the half and produced part together with double the square of half the given line.

The $\square^2$ of the whole produced line is = to the $\square^2$rs of half the given line and of the line made up of the half and produced part, together with twice the rectangle under the half and line made up of half and produced part. But double the rectangle under the half and line made up of half and produced part, together with the $\square^2$ of the produced part, is = to the $\square^2$ of the line made up of half and produced part together with the $\square^2$ of half the given line, (prop. 7, b. 2.). Therefore the $\square^2$ of the whole produced line together with the $\square^2$ of the produced part is = to double the $\square^2$ of half the given line and of the line made up of half and produced part.

Cor. The sum of the squares of two lines is equal to double the square of half their sum and double the square of half their difference. For the line made up of half and produced part is = to half the sum; and half the given line, to half their difference.

NOTE—The geometrical construction of this and the foregoing proposition being rather prolix and of little or no use, they are not necessary here; but in Dr. Elrington's Euclid they are very elegantly demonstrated.

Thus in numbers: suppose the given line is 8 feet and the produced part 2, the compound line will be 10 feet. Then $10^2 + 2^2$ (= 104) is = to $4^2 \times 2 + 6^2 \times 2 = 104$.

PROP. 11. PROB.

To divide a given finite right line so that the rectangle under the whole line and one segment, shall be equal to the square of the other segment.

From one extremity of the given line erect a perpendicular and make it = to the given line; bisect the perpendicular and connect the point of bisection with the other extremity of the given line; produce the perpendicular below the given line until the part compounded of the half and produced part is = to the connecting line, and cut from the given line a part = to the produced part; the $\square^2$ of this shall be = to the rectangle under the given line and the other segment.

Complete the $\square^2$ on the given line, and draw through the point of section a line parallel to the perpendicular; draw through the extremity of the produced part, to meet this parallel, a line par. to given line.

Then because the perpendicular is bisected and produced, the rectangle under the whole produced line and produced part, together with the $\square^2$ of the half line, is = to the $\square^2$ of the line made up of the half and produced part, (prop. 6, 2,); or to the $\square^2$ of the connecting line, which is = to the $\square^2$rs of given line and half perpendicular, (prop. 47, 1,); take away the $\square^2$ of the half line, then the rect. under the whole produced line and produced part is = to the $\square^2$ of the given line; if ∴ the rectangle under the perpendicular and produced part be taken away from both, (it being common to them,) the $\square^2$ of the produced or cut off segment shall be = to a rectangle under given line and the other segment.

Cor. The rectangle under the greater segment and difference between the segments is = to the square of the less segment, as is evident by taking the rectangle under the segments from the $\square^2$ of the greater segment and the rectangle under given line and lesser.

NOTE 1. Given the greater segment (AB) of a line, cut in extreme and mean ratio, to find the lesser.

Cut the given segment in extreme and mean ratio in C, and its greater segment AC shall be the segment required.

Fig. 1, plate 2.

Produce AB till BD is = to AC.

Then the rectangle under BA and AC or under AB and BD is with the $\square^2$ of AC or BD = to the $\square^2$ of AB; but the rectangle under AB and BD with the $\square^2$ of BD is = to the rectangle ADB; ∴ AC is the line required to be found.

2. Given the lesser segment of a line cut in extreme and mean ratio to find the greater.

Fig. 2.

Cut the given segment AB in extreme and mean ratio, in C; produce AB till BD is = to AC.

Then since AB is cut in extreme and mean ratio, the rectangle ABC or the $\square^2$ of AC (or of BD) with the rectangle BAC (or ABD) is = to the $\square^2$ of AB; AD ∴ is cut in extreme and mean ratio, and is evidently the line required to be found.

3. When a right line is cut in extreme and mean ratio, the square of the whole line is equal to the rectangle under the whole line and greater segment with the square of that segment.

For it is = to the rectangles under the whole and greater, and whole and less; but the rectangle under the whole and less is = to the $\square^2$ of the greater; ∴, &c. &c.

4. The square of the whole line is also equal to two squares of the greater, with a rectangle under the segments.

For one $\square^2$ of the greater with a rectangle under the segments is = to a rectangle under the whole and greater; ∴, &c.

5. The square of the whole line is also equal to three rectangles under the segments with two squares of the less.

For it is = to two rectangles under the segments with their $\square^2$rs; but the $\square^2$ of the greater is = to a rectangle under the segments with a $\square^2$ of the less; ∴ &c. &c.

6. The rectangle under the segments is equal to twice the rectangle under the whole line and intermediate part.

For the rectangle under the segments is = to a $\square^2$ of the less + a rect. under the less and difference; and the $\square^2$ of the less is = to a rect. under the greater and difference; ∴ the rect. under the segments is = to a rect. under the sum and difference, or to two rectangles under the sum and half diff. or intermediate.

7. The square of the line compounded of half the greater segment and the lesser segment is = to five times the square of half the greater segment.

It is evident that the $\square^2$ of half the greater segment is the fourth part of the rect. under the whole line and lesser seg.; but the rectangle under the whole line and lesser seg. together with the square of half the greater seg. is = to the square of the line made up of the half and lesser seg. (prop. 6, b. 2,); ∴ the $\square^2$ of half the greater is the fifth part of the $\square^2$ of this compound line.

8. The squares of the whole line and lesser segment are together equal to three times the square of the greater, or to three times the rectangle under the whole and lesser.

For the $\square^2$ of the whole line is = to a $\square^2$ of the lesser and two rectangles under the lines; if ∴ you add a $\square^2$ of the less to both, the $\square^2$rs of the sum and less will be = to a $\square^2$ of the greater and two rectangles under the sum and less, or two $\square^2$rs of the greater; ∴ &c. &c.

9. Therefore it is evident that the $\square^2$ of the sum of the whole line and lesser segment is = to five rectangles under them or to five $\square^2$rs of the greater segment.

PROP. 12. THEOR.

In an obtuse angled triangle, the square of the side subtending the obtuse angle, exceeds the sum of the squares of the sides, which contain the obtuse angle, by double the rectangle under either of these sides and the external segment between the obtuse angle and the perpendicular let fall from the opposite angle.

The $\square^2$ of the side subtending the obtuse angle is = to the $\square^2$rs of the perpendicular and line made up of the side (on which per. falls) and external segment; but the $\square^2$ of this line is = to a $\square^2$ of the side, a $\square^2$ of the seg. and two rectangles under them; and the $\square^2$ of the other side is = to the $\square^2$rs of the per. and seg.; ∴ the $\square^2$ of the side subtending the obtuse angle is = to the sum of the $\square^2$rs of the other two sides and two rectangles under the side, on which the perpendicular falls, and external segment.

NOTE.—If the triangle be isosceles, the square of the side subtending the obtuse angle is equal to two rectangles under the whole produced line and either side.

For it is = to the sum of the $\square^2$rs of the sides and two rect. under the side and ext. seg.; but those are together = to two rectangles under the whole produced line and either side.

PROP. 13. THEOR.

In any triangle the square of the side subtending an acute angle, is less than the sum of the squares of the sides containing that angle, by twice the rectangle under either of them and the segment between the acute angle and the perpendicular let fall from the opposite angle.

The $\square^2$ of the side, on which the perpendicular falls, with the $\square^2$ of the intercept, is = to two rectangles under them with the $\square^2$ of the other segment; if the $\square^2$ of the perpendicular be added to both, the $\square^2$rs of the side, intercept and perpendicular (or the $\square^2$rs of the sides containing the acute angle) are = to the double rectangle with the $\square^2$ of the other segment and $\square^2$ of the perpendicular or the $\square^2$ of the side opposite to the acute angle; ∴ this $\square^2$ with double the rectangle under the side on which the perpendicular falls, and intercept, is = to the sum of the $\square^2$rs of the other two sides.

Schol. 1. If the triangle be right angled, the perpendicular coincides with the side subtending the acute angle, and the rectangle under the base and intercept is the $\square^2$ of the base; but it is evident, that in this case the $\square^2$ of the side subtending the acute angle is less than the $\square^2$rs of the sides containing it, by twice the $\square^2$ of the base.

Schol. 2. Hence given in numbers the sides of any triangle, we can find its area; for subtract the $\square^2$ of one side, which is not the greater, from the sum of the $\square^2$rs of the other sides, and divide half the remainder, (that is, the rectangle under the side, on which the perpendicular falls, and the intercept,) by either of these sides, and subtract the $\square^2$ of the quotient from the $\square^2$ of the remaining side; the square root of the remainder, multiplied into half the former divisor, gives the area of the triangle.

Cor. If from any angle of a triangle, a right line be drawn bisecting the opposite side, the squares of the sides containing the angle is double the squares of the bisecting line and of the side subtending the angle.

If the bisecting line be perpendicular to the side, it is evident, from prop. 47, 1.

If not, let fall a perpendicular; then the $\square^2$ of one of the sides containing the angle, will be greater than the $\square^2$rs of the perpendicular and half side by a double rectangle, by which the $\square^2$ of the other side is less than the same $\square^2$rs: $\therefore$ &c.

NOTE 1. If the sides BA and AC containing the acute angle be equal, the square of the side BC subtending it is equal to two rectangles under either side A and the intercept DC.

Fig. 3.

For, BC^2 is $=$ to the sum of the $\square^2$rs of BA and AC $-$ two rectangles under CA and AD; but those two rectangles $+$ two rectangles under AC and CD are $=$ to two $\square^2$rs of either side AC or AB; $\therefore$ &c. &c.

2. If in the legs AC and AD of an acute angle, any two points D and C be assumed, and perpendiculars be let fall from them on the opposite sides, the rectangles under the assumed parts and their respective intercepts between the acute angle and perpendiculars are equal.

Fig. 4.

Join CD; then the $\square^2$ of CD is less than the sum of the $\square^2$rs of CA and AD by two rectangles under DA and AB, and also by two rectangles under CA and AE; $\therefore$ those rectangles are $=$.

3. If an obtuse DAC and an acute angle BAC be contained by lines DA and AC, and BA and AC respectively equal, and the excess of the one above a right angle be equal to the excess of a right angle above the

other, the square of the side DC subtending the obtuse angle exceeds the square of BC subtending the acute by four rectangles under either of them A C and the intercept between the perpendicular and angle.

Fig. 5.

Let fall the perpendiculars DE and BF, then EA is evidently = to AF, and the $\square^2$ of DC is = to the sum of the $\square^2$rs of DA and AC with two rectangles under CA and AE; and the $\square^2$ of BC is less than the $\square^2$rs of DA and AC by two rectangles CAE; ∴ BC^2 is less than DC^2 by four rectangles under CA and AE or CA and AF.

Cor. 2. Hence it is evident the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the sides.

PROP. 14. PROB.

To describe a square equal to a given rectilineal figure.

Make a rectangle = to it; if its adjacent sides be =, the problem is done.

If not, produce either side, and make the produced part = to the adjacent side; bisect the whole produced side, and describe a semicircle on it, and produce the other side to meet the circumference; this produced part is the side of the square sought.

For connect the point of bisection with its extremity at the periphery; the square of the intercept is = to the difference between the rs of the connecting line and intermediate part of the diameter, (prop. 47, 1,); but this dif. of $\square^2$rs is = to the rectangle under the unequal parts of the whole produced side (prop. 5, 2,) i. e. to the rectangle made = to the given figure.

END OF SECOND BOOK

THE
THIRD BOOK,
IN
GENERAL TERMS.

PROP. 1. PROB.

To find the centre of a given circle.

Draw any right line within the circle; let it be terminated by the circumference; bisect it, and from the point of bisection draw a line at right angles to it, and produce this line to meet the circumference; bisect it; its point of bisection shall be the centre of the circle.

For, if it be possible, let any other point be the centre of the given circle, the right lines drawn from it to the extremity of the assumed line are = to one another; also the segments of this line are = and the line drawn to its point of bisection common; ∴ the triangles formed by those are = in every respect; ∴ the angles made by the line drawn from the assumed point to the point of bisection are right; then one of them is = to an angle made by the perpendicular on this line, of which it is a part, which is absurd; ∴ this assumed point is not the centre; in like manner it can be proved, that no other, but the point of bisection of the line drawn at right angles from the point of bisection of the first drawn line, can be the centre of the circle.

Schol. Hence it is evident, that if any line terminated in a circle be bisected by a perpendicular, that perpendicular, if produced, shall pass through the centre.

PROP. 2. THEOR.

If any two points be taken in the circumference of a circle, the right line, which joins them, falls within the circle.

For, if possible, let the right line joining them be such, that it has a point outside the circle; connect the centre of the circle with its extremities and with this point.

Then in the triangle of which the line connecting the assumed points is the base, the angles at the base are =; but one of the angles made by the line drawn from the centre to the point outside the circle, being external to one of those, is greater than it, and ∴ than the other; ∴ the side opposite this angle, viz. the line drawn to the extremity from the centre, is greater than this drawn to the point outside the circle; but this drawn to the extremity is a radius of the circle, and this drawn to the point outside it greater than a radius; ∴ it is not less than the line to the extremity; ∴ this line connecting the assumed points is not a right line; in like manner it can be demonstrated that, if any other point but the extremities be in the circumference, it cannot be a right line; ∴ every point of the right line falls within the circle.

PROP. 3. THEOR.

If a right line drawn through the centre of a circle, bisect a right line which does not pass through the centre, it is perpendicular to it. And, if it cuts it at right angles it bisects it.

Part 1. Connect the centre of the circle with the extremities of the bisected line; then there are two triangles formed, having the half line and connecting line in one respectively = to the other half line and connecting line in the other, and the intercept between the centre and point of bisection common; ∴ (prop. 8, 1,) the angles at the point of bisection are =, and are ∴ right angles (prop. 13, 1.).

Part 2. Because the triangle formed by the line not passing through the centre, and the lines from the centre to its extremity is isosceles, the angles at this line are = and also those at the point of bisection (by constr.)

and the lines from the centre (viz. those opposite those = angles as being radii of the same circle) are = ; ∴ the two triangles are =, and also their other sides, viz. the parts of the line not passing through the centre ; ∴ this line is bisected.

PROP. 4. THEOR.

If in a circle two right angles cut one another, which do not both pass through the centre, they do not bisect one another.

If one of them pass through the centre, it is evident that it cannot be bisected by the other, which does not pass through the centre.

But if neither of the lines passes through the centre, draw a right line from the centre to their intersection ; if one of them be bisected by this, it is at right angles to it; and if the other be bisected by it, it is also at right angles to it ; ∴ an angle made by this line from the centre with one of the given lines is = to an angle made by it with the other which is a part of it, which is absurd ; ∴ the lines do not bisect one another.

PROP. 5. THEOR.

If two circles cut one another, they have not the same centre.

For, if possible, let them have a common centre, and from it draw two right lines, one cutting both circumferences, and the other to a point of bisection.

Then, since those lines are radii of one circle, they are =, and one of them drawn to the intersection is = to part of the other, (viz. of the one cutting the circles,) as being radii of the same circle, (def. 15, b. 1) ; ∴ this part is = to the whole line cutting the circles, which is absurd ; ∴ this point is not the centre of both circles ; and in the same manner it can be proved that no other point is the centre of both.

PROP. 6. THEOR.

If two circles touch one another internally, they have not the same centre.

This proposition may be proved exactly as the former, (from def. 15, b. 1.).

PROP. 7. THEOR.

If from any point within a circle, which is not the centre, right lines be drawn to the circumference, the greatest is that which passes through the centre.

The remaining part of the diameter is the least.

Those lines, which make equal angles with the diameter, are equal.

That line, which is nearer to that passing through the centre, is greater than one more remote.

And more than two right lines cannot be drawn, which shall be equal.

PART 1. The line passing through the centre is greater than any other drawn from this point.

Draw a right line from the centre to the extremity of the line not passing through the centre. This drawn line with the intercept between the centre and given point, are together greater than the line not passing through the centre (prop. 20, b. 1,); but this drawn line is = to the remaining part of the line passing through the centre (def. 15, 1,); ∴ the line passing through the centre is greater than the other, &c. &c.

PART 2. The other part of the diameter is less than any other line drawn from this point.

Draw a right line from the centre to the extremity of the line that is not a part of the diameter. Then this drawn line is less than the sum of the intercept between the centre and given point and the other line; but the intercept and remaining part of the diameter are together = to the connecting line, (def. 15, b. 1,); ∴ this remaining part of the diameter is less than any other line drawn from the given point.

PART 3. The right lines, which, at the same point, make = angles with the diameter, are =.

For, if possible, let one of them be greater than the other; cut from it (adjacent to the given point) a part = to the other; join the centre and the extremities of those = lines; produce to the circumference the line joining the centre and cut off part.

Then there are two triangles formed, having the cut off part and intercept in one respectively = to the other given line and intercept in the other, and the contained angles = (by hypoth.); ∴ the third sides are =, viz. the joining lines; but one of these is also = to the other produced to the circumference, (def. 15, 1,); ∴ a part of the produced line is = to the whole, which is absurd; ∴ those lines making = angles are not unequal.

Part 4. The line, which is nearer to that passing through the centre, is greater than the one more remote.

If the given lines be at the same side of the line passing through the centre, join the centre and their extremities; then there are two triangles formed, having two sides of the one, viz. the intercept and connecting line, respectively = to two sides of the other, viz. the intercept and the other connecting line; but in those triangles the angle subtended by the line nearest to that passing through the centre, is greater than that subtended by the other; ∴ (prop. 24, b. 1,) the side subtending the greater angle, viz. that nearest to the line passing through the centre, is greater than that subtending the lesser angle, viz. the more remote.

But, if the given lines be at different sides, draw from the given point a right line, cutting off with the diameter from the angle made by the remoter line with said diameter, an angle = to that which the line at the other side makes with it; this line shall be = to the given line on the other side, and is evidently greater than the other by the foregoing demonstration.

Part 5. More than two right lines cannot be drawn, which shall be equal.

For, let any three right lines be drawn from the given point to the circumference, and either one of them shall be part of the diameter, and ∴ greater or less than either of the others (by part 1 and 2); or, two of them must be at the same side of the diameter, and ∴ unequal (by part 4).

PROP. 8. THEOR.

If from any point without a circle, lines be drawn to the circumference, those which make equal angles with the line passing through the centre, are equal.

Of those lines, which are incident upon the concave cir-

cumference, the greatest is that which passes through the centre.

Of the rest, that, which is nearest to the line passing through the centre, is greater than the more remote.

But of those incident upon the convex circumference, that line is the least, which, if produced, would pass through the centre.

Of the rest, that, which is nearest to the least, is less than the more remote.

Only two lines can be drawn, either to the concave or convex circumference, which shall be equal.

PART 1. The right lines, which, drawn from the given point, make = angles with that passing through the centre, are =.

For, if possible, let one of them be greater than the other; cut from it a part (adjacent to the given point) = to the less; join the centre and the extremities of those = parts.

Then there are two triangles formed, having two sides of the one respectively = to two sides of the other, and the contained angles = (by hypoth.); ∴ the third sides or joining lines are =; but of these, the one joining the centre and uncut side is = to the other when produced to the circumference (def. 15, b. 1); ∴ that joining the centre and cut line is = to itself when produced to the circumference, a part to the whole, which is absurd. Therefore neither of the given lines is greater than the other; ∴ they are =.

PART 2. Of those lines, which are incident upon the concave circumference, that line, which passes through the centre, is greater than any other.

Join the centre and the extremity of that not passing through the centre. Then this joining line is = to that part of the line passing through the centre, between the centre and circumference; add to both the remainder of the line passing through the centre; then the added part and joining line are together = to the whole line passing through the centre; but the former are greater than the line not passing through the centre; ∴ the line passing through the centre is greater than the other.

PART 3. The line which is nearer to the greater, is greater than the more remote.

If the given lines be at the same side of that passing through the centre, join the centre and their extremities at the circumference; then there are two triangles formed, having a joining line and common side in one respectively = to the other joining line and common side in the other; but of the angles contained by those = sides, that subtended by the line nearest to that passing through the centre, is greater than that subtended by the more remote; ∴ (by prop. 24, b. 1,) the line nearest to that passing through the centre, is greater than the more remote.

But, if the given lines be at different sides, make on either side, at the given point and with the line passing through the centre, an angle = to the angle made by the given line at the other side; the line making this angle, when produced to the concave circumference, shall be = to the line at the other side. Then the proposition may be proved as in the foregoing case.

Part 4. Of those two lines incident on the convex circumference, that line, which, if produced, would pass through the centre, is less than any other.

Produce to the centre that one, that would pass through the centre, and connect with the centre the extremity of the other which is at the circumference.

Then this connecting line is = to the produced part; but the connecting line with the connected line are together greater than the whole produced line, (prop. 20, b. 1,); ∴ if the = parts be taken away, the line, which, if produced, would pass through the centre, is less than any other.

Part 5. The line, which is nearer to the least, is less than the more remote.

If the given lines be at the same side of the least, join the centre and their extremities at the circumference. Then it is evident (from prop. 21, b. 1,) that the line next the least is less than the more remote, for the joining lines are =, (def. 15, b. 1.).

But, if the given lines be at different sides of the least, draw at one side to meet the convex circumference, a right line, making at the given point and with the least, an angle = to that made by the line at the other side, then proceed as in the foregoing case.

Part 6. Only two equal right lines can be drawn either to the concave or convex circumference.

If any three lines be drawn, either one of them shall pass through the centre, and ∴ be either greater or less than the others, (by part 3,), or two must be at the same side of the line passing through the centre, and ∴ unequal.

Schol. It is evident, that any right line drawn to the convex circumference, is less than any right line drawn to the concave; it follows that, if any three lines be drawn from a point without a circle to its circumference, only two of them can be equal.

Cor. Hence, and from part 5, prop. 7, it is evident, that there is no point except the centre, from which three equal right lines can be drawn to the circumference of a circle.

PROP. 9. THEOR.

If a point be taken within a circle, from which more than two equal right lines can be drawn to the circumference, that point is the centre of the circle.

For, if it were a point different from the centre, only two equal right lines could be drawn from it to the circumference.

PROP. 10. THEOR.

One circle cannot cut another in more than two points.

For, if it be possible, let it cut the other in three points; connect the centre of one of the circles with those points; the connecting lines are =, (def. 15, b. 1); but, as the circles intersect, they have not the same centre, (prop. 5, b. 3); ∴ this centre is not the centre of the other circle, and ∴, as three right lines are drawn from a point not the centre, they are not =; but it was shown before that they were =, which is absurd; the circles ∴ do not intersect in three points.

Schol. Hence it is evident, that one circle cannot meet another in more than two points.

PROP. 11. THEOR.

If two circles touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.

For, if possible, let the centres be so situated, that the right line connecting them will not pass through a point of contact; produce it both ways to meet the circumferences, and connect the point of contact with the centres.

Then in the triangle formed by those connecting lines, the line connecting the centres and that connecting the centre of the internal circle with a point of contact, are together greater than the line connecting the centre of the external circle with this point of contact, (prop. 20, b. 1); ∴ than the radius of the external circle which passes through the centre of the internal; let the line connecting the centres be taken from both, and the radius of the internal circle shall be greater than the remaining part of that of the external, of which it is virtually a part, which is absurd; ∴, &c. &c.

PROP. 12. THEOR.

If two circles touch one another externally, the right line joining their centres shall pass through a point of contact.

For, if possible, let the centres be so situated that the right line joining their centres may not pass through a point of contact; join the centres and a point of contact.

Then in the triangle formed by the three connecting lines, those drawn from the centre to this point of contact, are together greater than that joining the centres; but they are also = to part of it, which is absurd; ∴ &c. &c.

PROP. 13. THEOR.

One circle cannot touch another externally or internally in more points than one.

If possible let two circles touch one another internally in two points; join the centres and produce the joining line to one point of contact; draw right lines from the centres to the other supposed point of contact. Then the right lines drawn from the centre of the lesser circle to the points of contact are =, (def. 15, b. 1); add to both the right line joining the centres; then the right line joining the centres and a radius of the lesser circle are together = to a radius of the greater; but they are also greater than it, (prop. 20, b. 1,) which is absurd.

Next, if possible, let two circles touch one another externally in two points; draw a right line joining the centres, and passing through one point of contact, and draw right lines from the centres to the other point of contact.

This then will evidently involve an absurdity, from def. 15, b. 1, and prop. 20, b. 1.

Therefore there is no case in which two circles can touch one another in two points.

Schol. Hence it is evident, that a circle, touching a circle, never again meets it.

PROP. 14. THEOR.

In a circle equal right lines are equally distant from the centre. And right lines, which are equally distant from the centre, are equal.

Join the centre and either extremity of each given line, and draw lines from the centre at right angles to the given lines.

PART 1. Then because the given lines are = and the perpendiculars from the centre bisect them, (prop. 3, b. 3,) half of one of them is = to half the other; ∴ the □²rs of those halves are =; but the right lines from the centre to the extremities being also = by hyp. and def., their □²rs are =; ∴ the differences between these equal squares, and the equal squares of halves are equal, i. e. the squares of the perpendiculars, (prop. 47, b. 1); ∴ the □²rs of the perpendiculars are =, and ∴ the perpendiculars themselves, (cor. 2, prop. 47, b. 1.)

PART 2. Because the perpendiculars are =, their □²rs are =; and, because the lines from the centre to the extremities are =, their □²rs are =; ∴ the differences between these latter equal squares and the equal squares of the perpendiculars are equal, by 47, 1, i. e. the squares of halves; ∴ the halves themselves are =, and ∴ the given lines are =.

PROP. 15. THEOR.

The diameter is the greatest right line in a circle; and, of all others, that, which is nearer to the centre, is greater than the more remote.

Part 1. The diameter is greater than any other line.

For, connect the centre with both extremities of the line not passing through the centre; those connecting lines are together = to the diameter (def. 15, b. 1,); but they are greater than the line not passing through the centre (prop. 20, b. 1,); ∴, &c. &c.

Part 2. That, which is nearer to the centre, is greater than the more remote.

First, let the given lines be on the same side of that passing through the centre, and not intersecting one another; connect the centre with the extremities of each of them.

Then, in the triangles formed by the given lines and those connecting lines, the lines connecting the centre and extremities of the one nearest to the centre are = to those connecting the centre and extremities of the other; but the angle contained by the former is greater than that contained by the latter; ∴ that line subtending the greater angle, viz. that nearest to the one passing through the centre, is greater than the line subtending the smaller angle, viz. the more remote.

Let the given lines be at different sides, or let them intersect. Draw perpendiculars to them from the centre, and from the greater perpendicular cut off a part = to the less, and through the point of section draw a line at right angles to this perpendicular: this drawn line is = to that on which the lesser perpendicular falls (prop. 14, b. 3,); but it is greater than that, on which the lesser perpendicular falls (by the foregoing case); ∴, &c. &c.

PROP. 16. THEOR.

The right line, drawn from the extremity of the diameter of a circle perpendicular to it, falls without the circle.
And if any right line be drawn from any point within that perpendicular to the point of contact, it cuts the circle.

Part 1. For, if it be possible, let there be a right line, which meets it again and is perpendicular to the diameter; connect the centre and this point of contact.

Then, because in the triangle thus formed, two of the sides are =, being radii of the same circle; ∴ the angles at the points of contact are =; but one of them is a right angle, which is absurd; ∴, &c. &c.

PART 2. If possible, let a right line be drawn from a point between the perpendicular and circumference to the extremity of the diameter, which does not cut the circle.

The angle contained by this drawn line and the diameter, must be less than a right angle; draw from the centre a line at right angles to it; this line drawn at right angles is less than a radius of the circle (prop. 19, b. 1,); but it is also greater than it, which is absurd; ∴ &c. &c.

Schol. 1. Hence it is evident, that the line at right angles to the extremity of the diameter, *touches* the circle in that point, and that the tangent can only meet the circle in one point, and that, at each point of the circumference, there is only one tangent.

Schol. 2. It is also evident, that the right line, which makes at the extremity of the diameter an acute angle, however great, must meet the circle again.

Cor. 1. From this proposition is immediately deduced a method of drawing a tangent through any given point in the circumference of a circle; draw through the given point a diameter, and erect at the extremity of it a perpendicular.

Cor. 2. If the radius drawn to the given point in the circumference be produced beyond its extremity at the centre, and if there be taken in the produced part any number of points, and, from these points as centres, circles be described through the given point, it is evident, that each of those circles touch the tangent in that point; whence it follows, that, if a right line be drawn from the centre to another point in the tangent, that part, which lies between the circumference and tangent, is *infinitely divisable.*

PROP. 17. PROB.

From a given point without a given circle, to draw a right line, which shall be a tangent to the circle.

From the centre of the given circle as a centre, and the given point as distance, describe a circle; connect the centre and given point; from the point, where this connecting line cuts the given circle, draw a line at right angles to it, to meet the circumference of the described circle; connect the point, in which it meets it, with the centre, and connect the point, in which this connecting line cuts the given circumference, with the given point; this line shall be the required tangent.

For thus there are two triangles formed, having two sides of one respectively = to two sides of the other (viz. radii of the circles), and the contained angle common; ∴ the other sides and angles shall be respectively =; but of these, the angle contained by a radius of given circle, (viz. part of the line connecting the centre and given point,) and the perp. is a right angle (by constr.); ∴ the angle contained by the other radius and last drawn line is a right angle; but this line is drawn from the given point to the circumference of the given circle, and is ∴ a tangent to it.

Schol. It is evident, that there can be drawn two tangents from the given point, one at each side of the line connecting the centre and given point.

PROP. 18. THEOR.

If a right line be a tangent to a circle, the right line drawn from the centre to the point of contact is perpendicular to it.

For, if possible, let any other right line from the centre be at right angles to it; then in the triangle formed by those three lines, (viz. the tangent and two from the centre to meet it,) since the line from the centre to the point of contact subtends a right angle, it is greater than the other line from the centre to the tangent; but it is = to a part of it, (def. 15, b. 1,) which is absurd. Therefore it is evident, that no other line from the centre, except this to the point of contact, is at right angles to the tangent.

PROP. 19. THEOR.

If a right line be a tangent to a circle, the right line perpendicular to it from the point of contact, passes through the centre of the circle.

For, if possible, let the centre be without this line; join it and the point of contact; this joining line is at right angles to the tangent (prop. 18, b. 3,); ∴ the angle contained by it and tangent is = to that contained by the given line and tangent, a part to the whole, which is absurd; ∴ this point is not the centre; and in the same manner it can be proved, that no other point, outside the line at right angles to the tangent, can be the centre of that circle.

PROP. 20. THEOR.

The angle at the centre of a circle is double the angle at the circumference, when they have the same part of the circumference for their base.

1. Let one side of the angle at the circumference pass through the centre; then, because, in the triangle formed by two radii and the line not passing through the centre, those radii are =, the angles contained by each of them and the line not passing through the centre are =; but the angle at the centre is = to their sum (prop. 32, b. 1,); ∴ it is double of that at the circumference.

2. Let the angle at the centre fall within that at the circumference; connect their vertices, and produce the line connecting them to the circumference; then this produced line divides the angle at the centre, and also that at the circumference; and each of those at the centre is double of its respectively internal remote at the circumference (by 1st case); ∴, &c. &c.

3. Let a side about the angle at the circumference cut a side about that at the centre; connect their vertices, and produce the line, connecting them, to the circumference.

Then the angle (at the centre) contained by the produced part and cut side, is double of that (at the circumference) contained by the connecting line and the side conterminous with the cut side; and that at the centre contained by the produced part and uncut side, is double of that at the circumference contained by the connecting line and cutting side; if those latter angles be taken from the former, the remaining angle at the centre, viz. the given angle at it, shall be double of the remaining angle at the circumference, viz. the given angle at it; ∴, &c. &c.

PROP. 21. THEOR.

The angles in the same segment of a circle are equal.

1. Let the segment be greater than a semicircle; con-

nect the centre with the extremities of the base of those angles.

Then the angle at the centre formed by those connecting lines, is double of each of the given angles, (prop. 20, b. 3,); ∴ the given angles are = to one another. (Ax. 7.)

2. Let the segment be a semicircle, or less than a semicircle: draw a right line from the vertex of one angle through the centre to meet the opposite circumference, and connect the vertex of the other angle with the point, in which this line meets the circumference. Then those drawn lines divide the given angles into angles, that are respectively = to one another, being angles in segments greater than semicircles, having the same parts of the circumference as their bases; ∴ the angles, into which one of them is divided, taken together, are = to those, into which the other is divided, taken together; ∴ the given angles are =; ∴, &c. &c.

Cor. If two equal angles stand upon the same arc, and the vertex of one of them be in the circumference of the circle, the vertex of the other shall be in the same circumference.

For, if it be possible, let the vertex of that other angle fall within or without the circumference; connect the point, where a side about it or that side produced cuts the circumference, with the other extremity of the arc, on which the given angles stand.

Then, because the angle thus formed, stands on the same arc with the given angle at the circumference, it is = to it, and ∴ to the other given angle, to which it is external or internal, contrary to prop. 16, b. 1.

PROP. 22. THEOR.

The opposite angles of a quadrilateral inscribed in a circle, are together equal to two right angles.

For of either triangle, into which one of the diagonals resolves the quadrilateral, one angle is an angle of the quadrilateral, and the other two taken together are = to the opposite angle of the quadrilateral, since they are respectively = to the parts, into which the diagonal resolves it.

Cor. If any side of a quadrilateral figure inscribed in a circle be produced, the external angle is equal to the internal remote angle; for the sum of each of these, and the internal adjacent angle, is equal to two right angles.

NOTE 1. From this proposition it is evident, that the angle in any segment of a circle, is = to the sum of the angles contained by its subtending chord, and the right lines drawn from the extremities of that chord to any one point in the opposite segment.

2. It is also evident, that, if one side of any triangle inscribed in a circle be produced, the external angle is = to the angle in the segment at the other side of the line subtending its adjacent internal angle.

PROP. 23. THEOR.

Upon the same right line, and upon the same side of it, two similar segments of circles cannot be constructed which do not coincide.

For, if it be possible, let two similar segments be constructed, having a point in one of them without the other; connect this point with the extremities of the line on which they stand, and draw from the point, in which either of those connecting lines cuts the inner seg., a right line to the opposite extremity.

Then, since the segments are similar, the angles they contain are =, (def. 10, b. 3); but one of them (viz. that in the inner seg.) is greater than the other (viz. that at the point, which falls outside, (by prop. 16, b. 1,) which is absurd; ∴ no point in either of the given segments can fall outside the other; ∴ &c.

PROP. 24. THEOR.

Similar segments of circles, standing upon equal right lines, are equal.

For, if the = right lines be so applied to one another, that an extremity of one may fall on an extremity of the other, the other extremities must fall on one another; ∴ the right lines coincide, (ax. 10); ∴ the segments themselves coincide, and therefore are =.

PROP. 25. THEOR.

A segment of a circle being given to describe the circle, of which it is the segment.

From any point in the given segment inflect two right lines; bisect them, and draw through the points of bisection lines at right angles to them; produce those lines till they meet; the point, in which they do meet, is the centre.

Because one of the inflected right lines is bisected by a perpendicular, this perpendicular passes through the centre, (prop. 1, b. 3); likewise the other perpendicular passes through the centre; ∴ the centre must be the intersection of those perpendiculars.

PROP. 26. THEOR.

In equal circles, equal angles, whether they be at the centres or circumferences, stand upon equal arcs.

First, let the given angles be at the centres; draw from the extremities of the arcs, on which they stand in each, right lines to any point in the circumference, and connect the extremities of the arcs in each.

Then, because the angles at the centres are =, and the sides about them are respectively =, the lines on which they stand are =, (prop. 4, b. 1,) i. e. the lines connecting the extremities of the arcs; also the angles at the circumferences are =, (prop. 20, b. 3, & ax. 7); ∴ the segments that contain them are similar; but they stand on = right lines; ∴ those segments are =; take away those equals from the entire circles, and the remainders, (viz. the arcs on which they stand) are =.

In the same manner it can be proved, that the arcs are =, if the given angles at the circumferences are acute, by drawing right lines from the extremities of the arcs to the centre.

But if the given angles at the circumferences be right or obtuse, bisect them, and the halves of them are =; and it can be proved as above, that the arcs, upon which those halves stand, are =; whence it follows that the arcs, on which the given angles stand, are equal.

PROP. 27. THEOR.

In equal circles, the angles, which stand upon equal arcs are equal, whether they be at the centres or circumferences.

For, if it be possible, let one of them be greater; cut from it a part = to the other.

Then, because, in the given = circles, the cut off angle in one is = to the given angle in the other; ∴ the arcs on which they stand are =, (prop. 26, b. 3); but the arcs, on which the original angles stand, are also =; ∴ the arc, on which the cut off angle stands, is = to the arc, on which the angle, it is cut off from, stands, a part to the whole, which is absurd; ∴, &c.

Cor. From the preceding proposition, it is evident, that, in a circle, right lines, which intercept equal arcs, are parallel, because the alternate angles are equal, and *vice versâ.*

PROP. 28. THEOR.

In equal circles, equal right lines cut off equal arcs, the greater equal to the greater, and the less to the less.

If the equal right lines be diameters, the proposition is evident.

If not, in each circle draw from the centre right lines to the extremities of the given line.

Then, because the circles are =, their radii are = to one another, and the given lines are =; ∴ the angles at the centres are =, (prop 8, b. 1,) and ∴ the arcs, on which they stand, are =, (prop. 26, b. 3); take away those = arcs from the = circles, and the remaining arcs are =.

PROP. 29. THEOR.

In equal circles, the right lines, which subtend equal arcs, are equal.

If the equal arcs be semicircles, the proposition is evident.

But if not, draw from the centre in each right lines to the extremities of the given lines.

Then, because the arcs subtended by the given lines are =, the angles at the centres standing on those arcs are =; but in the triangles, to which those angles belong, the sides about one angle are respectively = to those about the other, as being radii of = circles; ∴ the lines subtending them are =; ∴, &c.

Schol. Whatever has been demonstrated in the preceding propositions of equal circles, is also true of the same circle.

PROP. 30. PROB.

To bisect a given arc.

Connect its extremities; bisect the connecting line, and draw from the point of bisection a line at right angles to it to meet the circumference; it bisects the arc.

For, connect the point, in which it meets the circumference, with the extremities of the bisected line.

Then in the triangles formed by those connecting lines, the half lines and common perpendicular; a half line and perpendicular in one are respectively = to the other half line and perpendicular in the other, and the contained angles are = being right; ∴ the lines, opposite to them or the connecting lines, are =; ∴ the arcs, which they subtend, are =; ∴, &c.

PROP. 31. THEOR.

In a circle, the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is an acute angle; and the angle in a segment less than a semicircle is obtuse.

Part 1. The angle in a semicircle is a right angle.

Draw a diameter connecting the extremities of the sides about the angle, and join the vertex of it with the centre.

Then this connecting line being = to each part of the diameter at each side of the centre, the sum of the angles contained by it and each side of the given angle, i. e. the angle in the given semicircle, is (by prop. 5, b. 1) = to the sum of the angles contained by the diameter and sides about the given angle; ∴ (by cor. 1, prop. 32, 1,) said angle in semicircle is right; ∴ &c.

PART 2. The angle in a segment, greater than a semicircle, is acute.

Draw a diameter through the extremity, (remote from the given angle,) of either side about it; connect the said extremities, and draw through the extremity of that line, which is not conterminous with the diameter, a right line connecting the adjacent extremity of the diameter with ; this connecting line forms with the diameter an angle = to the given angle, since they are in the same segment; but the angle contained by the subtense of the given angle and this connecting line is a right angle, (part 1,); ∴ the angle, contained by the diameter and the line from its extremity to the extremity of the side not conterminous with it, is less than a right angle; ∴ the given angle is less than a right angle.

PART 3. The angle, in a segment less than a semicircle, is obtuse.

Take in the opposite circumference any point, and connect it with the adjacent extremities of the given sides.

Then, in the quadrilateral figure thus formed, the given angle and the angle at the assumed point are = to two right angles, (prop. 22, b. 3,): but the angle at the assumed point is an acute angle, (part 2, præc.); ∴ the given angle is obtuse.

Cor. 1. Hence can be derived a method of drawing through the extremity of any right line a perpendicular to it; take any point without it, and, with this point as a centre, describe a circle passing through the proposed extremity and cutting the given line in any point; draw through this point a diameter; the right line connecting the proposed extremity with the remote extremity of the diameter is at right angles to the given line, because it is the angle in a semicircle.

Cor. 2. Hence also can be drawn a tangent to a circle from a given point without it: draw a right line from the given point to the centre of the circle; on this line describe a semicircle, and a right line drawn from the intersection of this semicircle with the given circle is the required tangent; for it is perpendicular to the radius drawn to the point, where it meets the circle, the angle in a semicircle being a right angle.

PROP. 32. THEOR.

If a right line be a tangent to a circle, and from the point of contact a right line be drawn cutting the circle, the angle made by this line with the tangent, is equal to any angle in the alternate segment of the circle.

If the secant should pass through the centre, it is evident, that the angles are =, for each of them is a right angle, (prop. 18 and 31, b. 3.).

But, if the secant make an acute angle with the tangent, in which case of course the alternate segment is greater than a semicircle, draw through the point of contact a perpendicular to the tangent; connect its extremity with that of the secant, viz. those extremities remote from the point of contact.

Then in the triangle thus formed, the sum of the angles adjacent to the perpendicular is = to a right angle, as the perpendicular passes through the centre; but one of them is the complement of the angle in the alternate segment, as it is also of the angle made by the secant and tangent; ∴ the angle in the alternate segment is equal to that formed by the secant and tangent, having thus a common complement.

Also the angle in the segment less than a semicircle is = to the alternate angle contained by the secant and tangent.

For it, with that in the other seg., is = to two right angles; and also the sum of those at the point of contact is = to two right angles; but one of those has, in the foregoing case, been proved = to the angle in the greater seg.; ∴ the other is = to that in the lesser seg.

PROP. 33. PROB.

On a given right line, to describe a segment of a circle, that shall contain an angle equal to a given rectilineal angle.

If the given angle be right, bisect the given line, and describe a circle (with half of it as radius,); the circle is divided by the given line into two semicircles; ∴ each of them contains an angle = to the given right angle, (prop. 31, b. 3.).

But, if the given angle be acute or obtuse, make with the given line at either extremity of it an angle = to the given angle; draw through the vertex of this angle a line at right angles to the drawn line, and draw, from the other extremity of the given line, a line making an angle with it = to the angle which the perpendicular makes with the given line; the point where this line meets the perpendicular is the centre of the required circle.

For this line is = to that part of the perpendicular between it and the right angle, since the angles contained by them and the given line are =; ∴ the circle, described with this as a centre, and either of those lines from it to the given line as radius, will pass through the extremities of the given line; and the line, making the given angle with it, is a tangent to this circle, since it is at right angles to its radius; ∴, &c.

Schol. In the same manner a circle can be described, which shall contain an angle = to a given angle.

PROP. 34. PROB.

To cut off from a given circle a segment, which shall contain an angle equal to a given angle.

Draw a tangent to the given circle, and draw through the point of contact a secant making an angle = to the given angle.

This secant evidently cuts off the required segment.

PROP. 35. THEOR.

If two right lines within a circle cut one another, the rectangle under the segments of one of them, is equal to the rectangle under the segments of the other.

1. If the given right lines pass through the centre, they are bisected in the point of intersection; ∴ the rectangles under their segments are the $\square^2$rs of their halves, and are ∴ =.

2. Let one of the given lines pass through the centre, and the other not; join the centre and the extremities of that not passing through it.

Then the rectangle under the segments of that not passing through the centre is = to the difference between the $\square^2$rs of the intercept (between the centre and intersection of given lines) and either of the connecting lines, (cor. prop. 6, b. 2,) i. e. to the difference between the □²rs of this intercept and half the line passing through the centre; but the rect. under the segments of the line passing through the centre is also = to the dif. between the □²rs of its half and intercept (prop. 5, b. 2,); ∴ the rectangles are =.

3. Let neither of them pass through the centre; draw through their intersection a diameter. Then (by part 2, præc.) the rect. under the segments of each of the given lines is = to the rectangle under the segments of this diameter; ∴, &c.

PROP. 36. THEOR.

If from a point outside a circle two right lines be drawn to it, one of which is a tangent to the circle and the other cuts it, the rectangle under the whole secant and the external segment is equal to the square of the tangent.

1. Let the secant pass through the centre, draw a right line from the centre to the point of contact. The □² of the tangent is = to the difference between the □²rs of the segment (between the centre and given point) and the radius (prop. 47, b. 1,); but the rectangle under the whole secant and external segment is = to the same difference, (prop. 6, b. 2,); ∴, &c.

2. If the secant does not pass through the centre, connect the centre with the given point, with the extremities of the given lines and with the point where the secant cuts the circumference.

Then the rect. under the whole secant and external segment is = to the dif. between the □²rs of the radius and of the line from the centre to the given point (cor. prop. 6, b. 2,); but the □² of the tangent is also = to the same difference; ∴, &c.

Cor. 1. Hence, if from any point outside a circle, two right lines be drawn cutting the circle, the rectangles under them and their external segments are =; for each of the rectangles is = to the □² of the tangent.

Cor. 2. If, from the same point, two tangents be drawn to a circle, they are =; for their squares are equal to the same rectangles.

PROP. 37. THEOR.

If, from a point outside a circle, two right lines be drawn, one cutting the circle and the other meeting it; and, if the rectangle under the secant and external segment be equal to the square of the line, which meets the circle, that line, which meets it, is a tangent.

Draw from the given point a tangent to the circle; and connect the centre with said point, with the point of contact, and with the extremity of the meeting line.

The $\square^2$ of the tangent is = to the rect. under the secant and external segment; but the $\square^2$ of the meeting line is = to the same rect.; ∴ these $\square^2$rs are =, and ∴ the lines themselves; then, in the triangles formed by those = lines, the radii, and the line from the centre to the given point, the three sides of the one are respectively = to those of the other; ∴ the angle contained by the tangent and radius is = to that contained by the meeting line and radius (prop. 8, b. 1,); ∴ the meeting line is a tangent (prop. 16, b. 1.).

OBSERVATIONS.

1. *To cut a given right line* (AB) *so that the rectangle under the parts shall be equal to any square less than that of half the given line.*

Fig. 6. ANALYSIS.

Suppose it done, and that the rectangle under AC and CB is = to any given $\square^2$ less than half the $\square^2$ of AB; bisect AB, and describe on it a semicircle; from the point C raise a perpendicular CD, and join DA and DB. Then DAB is a right angled triangle, and the rect. ACB is = to the $\square^2$ of CD (note, prop. b. 2,); ∴ CD^2 is = to the given $\square^2$, and ∴ CD is = to its side.

Therefore, describe on the given line a semicircle; from either extremity B draw a perpendicular BE = to a side of the given $\square^2$; draw ED parallel to BA, and from the point D, in which it meets the circumference, let fall a perpendicular DC; it evidently cuts the given line in the required point.

2. *To produce a given right line* (AC) *so that the rectangle under the whole produced line and added part may be equal to any given square.*

Fig. 7. ANALYSIS.

Suppose it done, and that CD is the produced part; bisect the given line in B; then, since AC is bisected in B, and produced to D, the rect. under AD and DC, with BC^2, is = to the $\square^2$ of BD; ∴ BC^2 is the difference between the given $\square^2$ and BD^2; ∴ from either extremity C of the given line raise a perpendicular, and inflect from the point of bisection B a right line BE = to BD; then EC is evidently = to the side of the given $\square^2$.

Therefore, draw from either extremity C of the given line a perpendicular CE = to a side of the given $\square^2$; bisect AC in B; join BE, and produce AC till BD is = to BE; the rect. under AD and DC is = to the given $\square^2$.

For, the rectangle ADC, with the $\square^2$ of BC, is = to the $\square^2$ of BD (prop. 6, b. 1, of Elr.); but the $\square^2$ of BC is the difference between the $\square^2$rs of BE (or BD) and CE; ∴ the rectangle ADC is = to the $\square^2$ of CE, and ∴ to the given $\square^2$.

3. *If in a triangle* (HGI) *the rectangle under the segments* (GK and KH) *of one side made by a perpendicular* (IK) *from the opposite angle, be equal to the square of that perpendicular, the angle* (GIH), *from which it is drawn, is a right angle.*

Fig. 8. ANALYSIS.

Suppose that the angle GIH is a right angle; then the $\square^2$ of GH is = to the sum of the $\square^2$rs of GI and IH; if ∴ this can be proved, the proposition is true.

Because the rect. GKH is = to the $\square^2$ of KI, and KI^2 is the difference between IH^2 and KH^2; ∴ the rect. GHK is = to the $\square^2$ of HI; also the rect. HGK is = to the $\square^2$ of GI; but those rectangles are together = to the $\square^2$ of GH (prop. 2, b. 2. of Elr.); ∴ GIH is a right angle; ∴, &c.

4. *To produce a given right line* (GK) *so that the rectangle under it and the produced part may be equal to any given square.*

Fig. 8. ANALYSIS.

Suppose it done, and that the rectangle GKH is = to the given quantity, KH is the produced part; draw KI at right angles to GH; join GI and HI. Then because the rectangle GKH is = to KI^2, the angle GIH is right (prop. 4, b. 2, dedu.).

Therefore draw from either extremity K of the given line, a perpendicular KI = to a side of the given $\square^2$; join GI; draw through I, IH at right angles to IG, and produce GK to meet it, KH is evidently the required produced part.

5. *Given a scalene triangle* (ABC) *to produce one side* (CB) *so that the rectangle under it and the produced part may be equal to the difference of the squares of the other two sides.*

Fig. 9. ANALYSIS.

Suppose it done, and that BD is the produced part; join AD; it is = to the side AC; for, if not, draw any other right line AG (= to it) to meet the produced part; then, because AC and AG are =, the rectangle under CB and BG is = to the dif. between the $\square^2$rs of AB and AC (cor. prop. 1, b. 2,); but the rect. CBD is also = to it, which is absurd ; $\therefore$ AG is not $=$ to AC, nor is there any other line but AD $=$ to it.

Therefore, from A as a centre, with the radius AC describe a circle and produce CB to meet its circumference ; the rect. CBD is $=$ to the difference between the $\square^2$rs of BA and AC, (cor. prop. 6, b. 1.)

6. *The square of the greater of any two lines, is equal to four rectangles under the less and difference, with a square of the difference between the less and difference.*

For the $\square^2$ of the greater is $=$ to a $\square^2$ of the less, a $\square^2$ of the difference, and two rectangles under the less and difference ; but the $\square^2$rs of the less and difference are together $=$ to two rectangles under the less and dif. with a $\square^2$ of the difference between the less and difference.

7. *The square of one side of a triangle cannot be equal to the sum of the squares of the other two, and two rectangles under them ; for if so, this side would be equal to the other two.*

8. *To cut a given line* (AB) *so that the sum of the squares of the unequal parts may be equal to any quantity greater than two squares of half, and less than the square of the given line.*

Fig. 10.

ANALYSIS.

Suppose it done, and that the $\square^2$rs of AD and DB are together $=$ to the given quantity ; bisect the given line at C ; then two $\square^2$rs of AC, with two $\square^2$rs of CD, are $=$ to the sum of the $\square^2$rs of AD and DB, (prop. 9, b. 2) ; $\therefore$ the $\square^2$ of CD is $=$ to half the difference between the given quantity and two $\square^2$rs of half the given line.

Therefore bisect the given line in C, and cut its half CB so that the $\square^2$ of the intermediate part shall be $=$ to half the difference between two $\square^2$rs of half and the given quantity.

Then it is evident, from prop. 9, b. 2, that the $\square^2$rs of AD and DB are together $=$ to the given quantity.

9. *By being given the hypotenuse and difference of the squares of the sides of a right angled triangle, we can find the sides.*

For the difference between the sum of the $\square^2$rs and difference of the $\square^2$rs is = to two $\square^2$rs of the lesser side.

10. *It can be proved by the sixth of the second book, that the rectangle under the sum and difference of two lines, is equal to the difference of their squares: or, that the rectangle under two lines, plus the square of half their difference, is equal to the square of half their sum.*

11. *The common enunciation of the 5th and 6th is, that the rectangle under any two lines is equal to the difference between the squares of half their sum and half their difference.*

12. *The squares of the diagonals of any parallelogram are equal to the sum of the squares of the sides.*

For, by the last scholium of prop. 13, b. 2, the $\square^2$rs of the four sides are together = to four times the $\square^2$ of half each diagonal, i. e. to the $\square^2$rs of the diagonals.

13. *The square of the difference, the sum of the squares, and the square of the sum of any two lines, are in arithmetical progression.*

For they have a common difference; the difference between the $\square^2$ of the difference and sum of the $\square^2$rs is two rectangles under the lines; and the difference between the sum of the $\square^2$rs and $\square^2$ of the sum is the same.

14. *Given the hypotenuse and difference of the sides of a right angled triangle to find it, and its area.*

The hypotenuse being given, the sum of the $\square^2$rs is given; ∴ the difference between the sum of the $\square^2$rs and $\square^2$ of the difference is given; which is = to two rectangles under the sides; add this to the sum of the $\square^2$rs and their product is the $\square^2$ of the sum; ∴ you have the sum of the sides, and their difference being given; ∴ you can find the sides and ∴ the triangle.

15. *Given the sum of the sides and hypotenuse to find the triangle.*

The sum of the sides being given, the $\square^2$ of the sum is given; and the hypotenuse being given, the difference between the $\square^2$ of the sum and the sum of the $\square^2$rs is given; ∴ the difference of the sides is given, and ∴ the sides themselves.

16. *Given of any triangle the base, the line bisecting the base, and sum of the sides to find the triangle.*

Take from the $\square^2$ of the sum two $\square^2$rs of half the base and two $\square^2$rs of the bisecting line, which are together = to the $\square^2$rs of the sides, (cor. to 13, b. 2,); the remainder is = to two rectangles under the sides; then you can cut the given sum so that the rect. under the parts shall be = to half this difference, (by obsr. 1. b. 2,), and ∴ you have the sides.

17. *Given the sum of the sides and sum of the perpendicular and hypotenuse to find the triangle.*

The $\square^2$ of the sum of hypotenuse and perpendicular exceeds the $\square^2$ of the other sum by a $\square^2$ of the perpendicular; ∴ you can find the perpendicular; ∴ the hypotenuse, and ∴ the triangle.

18. *The square of the sum of the hypotenuse and perpendicular, exceeds the square of the sum of the sides, by a square of the perpendicular.*

For the $\square^2$ of the sum of the hypotenuse and perpendicular is = to the sum of their $\square^2$rs and four times the area of the triangle; and the $\square^2$ of the other sum is = to the sum of their $\square^2$rs and four times the area of the triangle; take from both quantities the areas, and there remains on one side the sum of the $\square^2$rs of the hypotenuse and perpendicular, and on the other the sum of the $\square^2$rs of the sides; ∴, &c.

19. *Given the perimeter and sum of the squares of the sides of a right angled triangle to find it.*

The sum of the $\square^2$rs being given, the hypotenuse is given, and ∴ the sum of the sides; ∴ you can find the triangle, (obser. 15, b. 3.).

20. *In a right angled triangle, given one side* (AB) *and a segment* (CD) *of the hypotenuse made by a perpendicular, to find the triangle.*

Fig. 11. ANALYSIS.

Suppose it done, and that ABD is the given triangle. Then it is evident that the rectangle DCA, with the $\square^2$ of CA, is = to the $\square^2$ of AB the given side: ∴ the rectangle DAC is = to the $\square^2$ of the given side.

Therefore produce the given segment until the rect. under the whole produced line and produced part is = to the $\square^2$ of the given side (obser. 3, b. 2,); this whole produced line shall be the hypotenuse of the required triangle; ∴ you can find the triangle.

21. *By having the area and hypotenuse of a right angled triangle, it can be found.*

For, divide the area by half the hypotenuse, and you will have the perpendicular, and then you can find the triangle.

22. *Given the area and difference* (AB) *of the sides of a right angled triangle, to find it.*

Fig. 12. ANALYSIS.

Suppose that ACD is the required triangle, and AB = to the given difference, BC is = to CD; but the rectangle under AC and CD is = to twice the area of ACD; ∴ the rect. ACB is = to twice the given area.

If therefore you produce the given difference until the rect. ACB is = to twice the area, it is evident, that AC and CB are the sides of the required triangle.

23. *Given the area and sum of the sides of a right angled triangle, to find it.*

Cut the given sum so that the rect. under the parts may be = to double the given area (by ob. 1. b. 2.)

24. *Given the segments of the hypotenuse to find the sides.*

Find a $\square^2$ = to the rect. under their sum and lesser segment and you have the lesser side, and also a $\square^2$ =

to the rect. under their sum and greater seg. and you have the greater side.

25. *Given the difference of the sides and difference of their squares to find the sides.*

Produce the given difference so that the rectangle under it and produced part may be = to the difference of the $\square^2$rs (ob. 3, b. 2,); this produced part shall be = to the sum of the sides (prop. 5, b. 2,); ∴ having the sum and difference, you can find the sides.

26. *To divide a given right line* (AD) *so that the sum of the squares of the whole line and one part, may be equal to three squares of the other part.*

Fig. 1.

ANALYSIS.

Suppose it is cut in the point B as required; then the $\square^2$rs of AD and DB are = to three $\square^2$rs of AB; but the $\square^2$rs of AD and DB are together = to a $\square^2$ of AB, two $\square^2$rs of BD, and two rect. ABD; ∴ two rectangles ABD, with two $\square^2$rs of BD (i. e. two rect. ADB), are = to two $\square^2$rs of AB; ∴ the square of AB is = to a rect. ADB.

If then you cut the given line by prop. 11, b. 2, the $\square^2$rs of the whole line and lesser segment will be together = to three $\square^2$rs of the greater segment.

27. *In a scalene triangle, the square of the greater side, the sum of the squares of the bisecting line and half the base and the square of the lesser, are in arithmetical progression.*

For they have a common difference, viz. double the rectangle under half the base and intercept between the perpendicular and bisector.

28. *Given two sides of a triangle and a line bisecting the third side to find the area.*

Take double the $\square^2$ of the bisecting line from the sum of the $\square^2$rs of the sides, and there will remain two $\square^2$rs of half the third side; ∴ this side can be found, and ∴ the triangle.

29. *The difference of the segments made by a perpendicular from any angle of a triangle, on the opposite side, must be greater than the difference of the sides.*

For the rectangle under the sum and difference of the sides, is $=$ to a rectangle under the sum and difference of the segments, since the difference of their $\square^2$rs are $=$, (cor. prop. 47, b. 1) ; $\therefore$ it is evident, &c. &c.

30. *The square of the hypotenuse of a right angled triangle, plus four times the area of the triangle, is equal to the square of the sum of the sides.*

For the $\square^2$ of the sum is $=$ to the $\square^2$ of the hypotenuse and two rectangles under the sides; but two rect. under the sides are $=$ to four times the area of the triangle ; $\therefore$, &c. &c.

31. *Given one side of a right angled triangle, and the difference between the hypotenuse and the other side, to find it.*

The $\square^2$ of the given side is $=$ to a rectangle under the sum and difference of the hypotenuse and third side, (prop. 5, b. 2) ; for it is $=$ to the difference of their $\square^2$rs, (cor. prop. 47, b. 1) ; then produce the given difference, so that the rectangle under it and produced part may be $=$ to the $\square^2$ of the given side (by obser. 3, b. 2 ;) take half the produced part and add to it half the given difference ; their sum shall be the hypotenuse ; and the difference between half the difference and half the production will be the third side.

32. *Given the rectangle under any two lines and their difference to find them.*

Fig. 6.

Take CF $=$ to half the given difference ; from its extremity C erect a perpendicular CD $=$ to the side of a $\square^2$, which is $=$ to the given rectangle ; join CF ; and from F as centre with the radius FD describe a circle, and produce FC both ways to meet it in A and B ; it is evident, that AC and CB are the lines required to be found.

33. *If the sides about an obtuse angle be produced, and perpendiculars be drawn to them from the opposite angle, it is evident from prop.* 12, *b.* 2, *that the greatest intercept is conterminous with the least side.*

34. *If a right line be drawn, cutting the circumferences of two concentric* circles, the intercepts are equal.*

If it passes through the centre, it is evident; but, if not, let fall a perpendicular on it from the centre; then it is evident, that the entire drawn line is bisected by it, and also that part of it within the inner circle; if ∴ this latter part be taken away, the remainders, viz. the intercepts, shall be =.

* Concentric circles are those, which, lying one within another, have a common centre.

35. *To draw a chord in a circle, whose square shall be equal to half the square of the diameter.*

Fig. 13.

Bisect the semicircular arc AB by CD; join DB; the $\square^2$ of DB is = to half the $\square^2$ of AB.

For it is = to the sum of the $\square^2$rs of BC and CD, and they are together = to half the $\square^2$ of AB, (cor. prop. 4, b. 2.).

36. *The converse of the second part of prop.* 15, *b.* 3, *is added by Robert Simpson, which is as follows: And the greater line is nearer to the centre than the less, the demonstration of which is evident.*

37. *It is evident from prop.* 22, *b.* 3, *that any external angle of a triangle, is equal to the angle in the remote segment, whose base is the subtense of the internal angle adjacent to the external.*

For either of them with this adjacent internal is = to two right angles.

38. *If two diameters in a circle intersect one another at right angles, they divide the circumference into four equal arcs, (prop.* 26, *b.* 3)*; and by drawing the subtenses of the right angles, a square will be inscribed in the circle.*

If those right angles be bisected, and the subtenses of the halves be drawn, an octagon will be inscribed; and if trisected, a duodecagon.

By trisecting the right angles also, an hexagon may be inscribed, and also an equilateral triangle.

For suppose the circumference to be 360 degrees, by trisecting the right angles, it is divided into arcs of 60° and 120°; $\therefore$ by connecting the subtenses of those of 60°, an hexagon will be formed, and the sides of the hexagon are evidently $=$ to the radii of the circle.

By connecting the subtenses of those of 120°, an equilateral triangle will be formed.

It is evident, that the $\square^2$ of the side of an equilateral triangle inscribed in a circle is $=$ to three times the $\square^2$ of the radius.

For it is the difference between the $\square^2$ of the diameter bisecting one of the angles of the equilateral, and the $\square^2$ of the line connecting the extremity of the diameter and base; but that connecting line $=$ radius; $\therefore$, &c.

39. *If two parallel chords be drawn in a circle, they intercept equal arcs, (prop.* 26, *b.* 3.*).*

40. *In prop.* 35, *b.* 3, *if right lines be drawn, joining the extremities of the sides of the angles vertically opposite, there shall be formed two similar triangles.*

This is evident, from prop. 32, b. 1, and prop. 21, b. 3.

41. *To find the area of an equilateral triangle, by having the difference of the perpendicular and half side.*

Take half the given difference, and extract the root of three times its $\square^2$, to which root add the same half and multiply their sum by the difference and said sum. (See prop. 115, b. 1.) For finding the side, take the $\square^2$ of half the excess from the $\square^2$ of the excess, and extract the root of the difference; this root $+$ half the excess is $=$ half the side, (prop. 115, b. 1.).

A VARIETY

OF

THE MOST IMPORTANT

PROBLEMS AND THEOREMS,

DEDUCIBLE

FROM THE SECOND AND THIRD BOOKS,

DONE ANALYTICALLY.

PROP. 1. THEOR.

The difference between the hypotenuse (BC) *and sum of the sides* (BA and AC) *of a right angled triangle* (ABC) *is equal to two radii of the inscribed circle.*

Fig. 14. ANALYSIS.

Suppose it to be the case; then since BC together with FG and GE are = to BA and AC together, BC with FA and AE are = to BA and AC; for it is evident that FA and AE are respectively = to GE and FG, since each of the angles F, A, and E, is a right angle; ∴ BC is = to BF and CE together; if this can be proved, the proposition is evident.

Join BG; then, since, in the triangles BGD and BGF, the sides BG and GD are respectively = to BG and GF, and the angles BDG and BFG are = as being right, and the angles DBG and FBG of the same affection, the triangles are = in every respect; ∴ BD is = to BF; for the same reason CD is = to CE; ∴ BC is = to the sum of BF and CE; ∴, &c.

PROP. 2. PROB.

To produce a given right line (AB) *so that the rectangle under the whole and produced part may be equal to a given quantity.*

Fig. 15.

ANALYSIS.

Suppose it done, and that AC is the line produced; let F be = to a side of the $\square^2$ of the given quantity; then it is evident, from prop. 37, b. 3, that CA shall be = to the secant drawn from the extremity of the tangent (= to F), through the centre of the circle, the diameter of which is = to the given line.

Describe a circle, of which the diameter is the given line AB; draw to it a tangent IA = to the given quantity, and from its extremity I draw through the centre the secant ID; it is evidently = to the whole produced line; ∴ make BC = to GI, and the rect. ACB is = to the given quantity.

Cor. 1. By being given the tangent and internal part of the secant passing in any way, we can find the secant.

Cor. 2. By being given the tangent and external part of the secant, we can find the secant.

Take the $\square^2$ of the given part from the $\square^2$ of the tangent, and then produce the given part so that the rectangle under the parts may be = to the remainder; this produced line shall be the required tangent.

PROP. 3. THEOR.

The perpendicular (AI) *of an equilateral triangle* (ABC) *is equal to three times the radius of the inscribed circle.*

Fig. 16.

It can be proved, that each of the radii, from the centre to the points of contact, bisects the side, to which it is drawn (prop. 32, b. 1,); and, if they be produced, they will bisect the angles; also produce DE until EF is = to it, and draw FC; then the triangles ADE and FEC are evidently =, and have also their sides and angles respectively = to one another; and also the triangles CDE and CEF are = in every way; ∴ the angle FCD is the angle of an equilateral triangle; and since the angles FDC and DFC are =, they must be the angles of an equilat.; ∴ DF is = to FC or DA; but DF is double of DE or DI; ∴ DA is double of DI or DE; ∴ AI is three times DI or DE; ∴, &c.

PROP. 4. PROB.

Given of any triangle the base, line bisecting the base, and sum of the sides, to find the triangle.

Take from the $\square^2$ of the sum two $\square^2$rs of half the base, plus two $\square^2$rs of the bisecting line; take half the remainder and find the side of a $\square^2$ = to it; then cut the given sum so that the rectangle under its segments may be = to this $\square^2$, &c.

PROP. 5. THEOR.

If a tangent (FH) *be drawn to a circle, parallel to a chord* (AB), *the point of contact* (F) *will be the middle point of the arc cut off by that chord.*

Fig. 13. ANALYSIS.

Suppose it to be the case; join FA and FB; then, since F is the middle point of the arc, FA and FB are =; ∴ the angles FAB and FBA are =; if this can be proved, the proposition is true.

Since FH is parallel to AB, the angle ABF is = to BFH; but BFH is = to BAF, (prop. 32, b. 3,); ∴ ABF is = to BAF; ∴ FA is = to FB, &c.

PROP. 6. PROB.

To draw a common tangent, to two given circles, in a transverse direction.

Fig. 17. ANALYSIS.

Suppose it done, and that OL is the required tangent; join FO and LE, (i. e. the centres with the points of contact); produce FO till OH is = to LE; join HE; because OL is a tangent, the angles FOL and ELO are right; ∴ FH is parallel to LE; and, since OH is = to LE, HE is parallel to OL; ∴ the angle FHE is a right angle.

Then join the centres F and E; and on FE describe a semicircle FHE; in it inflect a right line FH = to the sum of the radii FO and LE; join HE; draw EL parallel to HF and join LO: LO is the tangent; the demonstration is evident.

Also a direct tangent can be drawn to two given circles.

1. If the given circles be =, connect their centres; draw a radius in each on the same side and at right angles to the connecting line, and connect their extremities at the circumference; this connecting line is the required tangent.

For, the circles being =, the radii are =; and they being =, and at right angles to the line connecting the centres, they are parallel; ∴ the connecting lines are = and par.; ∴ the radii are at right angles to the one at the circumferences; ∴ it is a tangent.

2. If the given circles be unequal:

Fig. 18. ANALYSIS.

Suppose TH is the required tangent, TI and HE are evidently par. to one another; make TF = to HE, and join FE; it is = and par. to TH; ∴ the angle IFE is right, and IF is the difference between TI and HE; with FI as radius, describe a circle FA; EF is evidently a tangent to it.

Therefore, with the centre I and the difference of the radii IF, describe a circle FA; and, from the centre E of the lesser circle, draw a tangent EF to it; produce IF to T; draw EH par. to IT, and join TH; it is evidently the required tangent.

PROP. 7. THEOR.

The distance from the centre of the chord (AB), *whose square is three fourths of the square of the diameter* (CD), *is half the radius.*

Fig. 19.

Bisect AB in F, and join EF, EA and EB.

AB^2 is three-fourth of CD^2; but AB^2 is = to four times AE^2 = four times EF^2; ∴ four times EF^2 must be = to EB^2, which is one-fourth of CD^2; ∴ EF is = to half of ED.

PROP. 8. PROB.

Given of any triangle, the base, vertical angle, and difference of the sides, to find it.

Fig. 20.

ANALYSIS.

Suppose it done, and that ADC is the required triangle, let AC be = to the given base, AB to given difference, and ADC to the given angle; join BC. Then, because AB is the difference between AD and CD, DB and DC are =; ∴ the angles DBC and DCB are =; ∴ DBC is half the supplement of the given angle, and ∴ ABC is the supplement of half the supplement of the given angle.

Therefore, on the given base describe a segment of a circle capable of containing the given angle, and also a segment capable of containing the supplement of half the supplement of the given angle; inflect in the latter segment, from one extremity of the given base, a right line = to the given difference; produce it to meet the periphery of the other segment, and join its extremity with the other extremity of the given base; thus is formed the required triangle. The demonstration is evident.

PROP. 9. THEOR.

If from any point (F) *in a rectangle* (ABCD), *four right lines be drawn to the four angles, the sums of the squares of those drawn to the opposite angles, will be equal*, (*i. e.* $FA^2 + FC^2 = FB^2 + FD^2$).

Fig. 21.

For let the diagonals AC and BD be drawn; they bisect each other in E, and join EF; then the triangles ABC and BAD being = in every way, (prop. 24, b. 1,): thence will AE, (i. e. half AC) = DE, i. e. half DB; but $FA^2 + FC^2 = 2\ AE^2$ (i. e. DE^2) $+ 2\ EF^2 = FB^2 + FD^2$.

PROP. 10. PROB.

Given the base (AB) *and sum of the squares of the sides of any triangle, to find the locus of the vertex.*

Fig. 22.

ANALYSIS.

Suppose it done, and that AEB and ADB are any number of triangles on the given base; the sum of the $\square^2$rs of their sides are =, as being each = to the given quantity; then it is evident, that the difference between the sum of the $\square^2$rs of AE and EB, and two $\square^2$rs of half

AB is the $\square^2$ of the bisecting line EC; also the difference between the sum of the $\square^2$rs of AD and DB, and two $\square^2$rs of half AB, is = to the $\square^2$ of the bisecting line DC; ∴ DC^2 is = to EC^2; ∴ CE is = to CD; ∴ the circle described with C as a centre and CE as radius, will pass through the vertices of any number of triangles on the given base, the sum of the $\square^2$rs of whose sides shall be =.

Therefore, from the sum of the $\square^2$rs take two $\square^2$rs of half AB, the remainder shall be = to two $\square^2$rs of the bisecting line; then the bisecting line can be found; and it is evident, that the required *locus* lies in the circumference of the circle of which this bisecting line is radius.

PROP. 11. THEOR.

In every quadrilateral figure (ABCD) *the sum of the squares of the sides is equal to the sum of the squares of the diagonals, plus four times the square of the line* (EF) *joining the points of bisection of the diagonals.*

Fig. 23.

Join EA and ED.

Then the sum of the $\square^2$rs of CA and AB is = to two $\square^2$rs of CE + two $\square^2$rs of EA; and the $\square^2$rs of CD and DB are = to two $\square^2$rs of CE + two $\square^2$rs of ED; or there are four $\square^2$rs of CE, which are together = to CB^2; and there are also two $\square^2$rs of AE and two $\square^2$rs of DE; and those are together = to AD^2 + four times FE, (prop. 13, b. 3.).

PROP. 12. THEOR.

If any number of equal right lines (EF, IL, and KT) *be placed in a circle, the locus of their points of bisection will be in the circumference of a circle.*

Fig. 24.

For, if right lines OK and OP be drawn from the centre O to the points of bisection K and P, they are evidently = to one another (prop. 14, b. 3, Elr.); ∴ the circle described with O as a centre and OK as radius, shall pass through the points of bisection of every right line placed within that circle.

PROP. 13. THEOR.

The locus of the centres of any number of circles, which shall touch two given right lines (AD and AB) *that are not parallel, is the right line* (AF) *bisecting the angle* (BAD) *made by the given lines.*

Fig. 25.

For, from any points I and T in AF, let fall perpendiculars IB and ID or TL and TS; it is evident, that IB and ID are = to one another, and ∴ the circle described with the centre I, and radius ID, touches the given lines; also the circle described from T as a centre and TL as radius, touches the given lines; ∴, &c.

Cor. From this proposition, by being given the radius, a circle can be described to touch two given lines, that are not parallel.

For, draw AP at right angles to AB, and make it = to the given radius, draw PI parallel to AB and draw IB par. to PA; IB is evidently = to the given radius; ∴, &c.

PROP. 14. PROB.

To draw from two given points (A and B) *on the same side of a given right line* (CD,) *two right lines, to the same point in the given line, which shall contain a given angle.*

Fig. 26.

ANALYSIS.

Suppose that AL and BL are the lines required to be drawn, and ALB = to the given angle, join AB, and describe through the points A, L, and B, a circle: then it is evident that the segment ALB is one capable of containing the given angle.

Therefore join AB; on it describe a segment of a circle capable of containing the given angle; either of the points where its circumference cuts the given line, is the point to which the lines are to be drawn.

PROP. 15. THEOR.

If a right line (CD) *be divided into five equal parts, the square of the line* (CB) *made up of four of those parts, plus the square of the line* (CA) *made up of three of them is equal to the square of the whole line.*

Fig. 27.

For, since CB is divided in A and AB = to BD, (prop. 8, b. 2, Elr.) the $\square^2$ of CD is = to $CA^2 + 4\ CB \times BA$; but $4\ CB \times BA$ is = to CB^2; ∴, &c.

PROP. 16. PROB.

Given the hypotenuse to describe a right angled triangle, so that the hypotenuse and one side may be together double of the third side.

Divide the given right line into five = parts; four of those parts shall be = to the greater side, and three to the lesser.

For, by adding the lesser to the given line, their sum shall be = to 8 of the parts, which is ∴ double of the other.

Cor. And those lines shall form a right angled triangle.

For the $\square^2$ of the given line is = to the sum of their $\square^2$rs, by the foregoing prop.

PROP. 17. PROB.

Given the hypotenuse of a right angled triangle, and the sum or difference of the base and perpendicular, to construct the triangle.

Fig. 28.

ANALYSIS.

Suppose it done, and that ABC is the required triangle: let AC be the given hypotenuse, and AD and AE = to the sum or difference of the base and perpendicular; BD or BE is = to BC; join CE and CD; then CED is evidently an isosceles right angled triangle; ∴ each of the angles CED and CDE is half a right angle; the points D or E being given, the right line EC or DC is given in position.

If therefore a right line DC or EC be drawn from either extremity of the sum or difference, making the angle CED or CDE = to half a right angle, and inflect from the other extremity A a right line AC (= to the hypotenuse) to meet EC or DC, and let fall from the point C in which they meet a perpendicular CB, ACB is evidently the required triangle.

PROP. 18. PROB.

To draw through a given point, (L) *either inside or outside a given circle* (ABC), *a right line, the part of which intercepted by the circle, shall be equal to a given right line, not greater than the diameter of the circle.*

Fig. 29.

ANALYSIS.

Suppose it done, and that LD is the line required to be drawn; let AB be the given line; draw from the centre I, IE and IF at right angles to them; IE and IF are $=$; $\therefore$ the circle described from I as centre, with IE as radius, will touch both lines in E and F.

Therefore let fall IE at right angles to the given line, with I as centre and IE as radius, describe a circle ETF; draw through the given point L a tangent LD to this circle: the part of it CD is evidently $=$ to AB.

PROP. 19. THEOR.

The squares of the diagonals of any parallelogram, are together equal to the sum of the squares of the sides.

Fig. 21.

For the diagonals evidently bisect one another, and the $\square^2$rs of one pair of sides (AB and BC) are together $=$ to $2\ AE^2 + 2\ EB^2$; and the $\square^2$rs of the other pair of sides are together $=$ to $2\ AE^2 + 2\ ED^2$; but $4\ AE^2 + 4\ EB^2$ are together $=$ to $AB^2 + BD^2$; $\therefore$, &c.

PROP. 20. THEOR.

If in a circle two chords (AB and CD) *cut one another at right angles, the sum of the squares of the segments is equal to the square of the diameter.*

Fig. 30.

Draw the diameter BF; join CF, FA and AD; then FB is $=$ to $FC^2 + CB^2$ (prop. 47, 1 Elr.); but CB^2 is $=$ to $CH^2 + HB^2$; and CF^2 is evidently $=$ to AD^2*; but AD^2 is $=$ to $AH^2 + HD^2$; $\therefore$ FB^2 is $=$ to $CH^2 + HD + BH^2 + HA^2$.

* CF is $=$ to AD; for CD is parallel to FA; join CA, and then it is evident.

From this also it is evident, that in a circle parallel chords intercept equal arcs.

PROP. 21. THEOR.

If from the centre (A) *of a given circle a right line be drawn to any point* (C) *in a chord, the square of this drawn line, together with the rectangle under the segments of the chord, is equal to a square of the radius.*

Fig. 31.

Produce AC both ways to L and F.

Then the rectangles LCF and DCB are $=$; but the rect. LCF with AC^2 is $=$ to AF^2; $\therefore$ the rect. DCB with AC^2 is $=$ to AF^2.

PROP. 22. THEOR.

If in a circle two chords (DB and EI) *cut a diameter* (LF) *at the same point and at equal angles, they are equal to one another.*

Fig. 31.

Bisect DB and EI in K and H; join AK and AH. Then, because the angles ACH and ACK are $=$ (by hypoth.), and the angles AKC and AHC $=$ (by prop. b. 3, Elr.), and AC common, AK is $=$ to AH, (prop. 26. b 1. Elr.); $\therefore$ DB is $=$ to IE (prop. b. 3. Elr.).

PROP. 23. PROB.

The diameter (AC) *of a circle being produced to a given point* (B) *to find in the produced part a point, from which, if a tangent be drawn to the circle, it shall be equal to the segment of the produced part between this point and the given point.*

Fig. 32.

ANALYSIS.

Suppose E is the required point, and EF the tangent, join BF and produce it to L; join L with the centre O, and join OF.

Then the angles LFO, OFE, and EFB are evidently together $=$ to the angles BLO, LOB, and OBL; and LFO is $=$ to FLO, and also EFB to EBF; $\therefore$ EFO is $=$ to LOB; $\therefore$ LOB is a right angle.

Then draw LO at right angles to BA; join BL and FO, and draw FE at right angles to FO; E is evidently the required point.

For the angles at F are together = to the angles at B, L, and O; but OFL is = to OLF; and EFO to BOL; ∴ the angle EBF is = EFB; ∴, &c. &c.

PROP. 24. PROB.

To describe a circle, which shall have a given radius, and its centre in a given right line (EF), *and shall touch another right line* (BF), *making a given angle with the former.*

Fig. 33.

ANALYSIS.

Suppose it done, and that CL is the circle required to be described; join D (the centre) and point of contact C; CD is = to the given radius and is at right angles to the given line BF.

Then inflect between BF and EF a right line CD = to the given radius, and making at the point C, in which it meets B, a right angle (prop. 27. on b. 1.).

It is evident, that the circle, described from D as a centre with DC as radius, touches BF (in C), has the given radius, and its centre D in EF.

PROP. 25. THEOR.

If the diagonals (IL and BD) *of a quadrilateral figure inscribed in a circle cut one another at right angles, the sum of the squares of the opposite sides are equal.*

Fig. 34.

For the $\square^2$rs of BL and ID are = to the $\square^2$rs of BF, FL, DF, and FI (prop. 47. b. 1. Elr.); also the $\square^2$rs of BI and LD are together = to them; ∴ $BL^2 + Dl^2$ is = to $BI^2 + LD^2$.

PROP. 26. THEOR.

If two points be taken in the diameter of a circle equally distant from the centre, the sum of the squares of the lines, drawn from these to any point in the circumference, is always the same.

For the sum of their $\square^2$rs is = to a $\square^2$ of the radius + a $\square^2$ of the distance from the centre to either of the given points.

PROP. 27. THEOR.

If from any two points (B and D) *in the circumference of the greater of two concentric circles, two right lines* (BA and DE,) *be drawn to touch the other circle, they shall be equal to one another.*

Fig. 34.

Let O be the centre; join OE, OA, OB, and OD.

Then in the triangles OAB and OED, the sides OA and OB are respectively = to OE and OD; the angles OAB and OED right angles, and the angles ODE and OBA of the same affection, the triangles are = in every respect (prop. 78 on b. 1.); ∴ BA is = to DE.

PROP. 28. THEOR.

The diagonals (BE and AC,) *of a regular pentagon cut one another in extreme and mean ratio.*

Fig. 35.

In the pentagon ABCDE, the diagonal BE may be proved parallel to DC, and ∴ EH = to EA, and ∴ to AB; and it can also be proved that AC is bisected by the right line BF drawn at right angles to ED; hence the rectangle AGC + BG^2 is = to AB^2 or EH^2; but the rectangle AGC is = to the rect. AHC (= to the rect. EHB); hence the rect. EHB + HB^2 is = to EH^2, i. e. the rect. EBH is = to EH^2; ∴, &c.

PROP. 29. PROB.

Given the area of a rectangle, and the excess of one side above another to construct it.

Fig. 36. ANALYSIS.

Suppose that DA and EC are = to the sides of the required rectangle, and ED = to the difference; then it is evident, that AD is the secant passing through the centre of a circle, to which the side of a □², = to the given area, is a tangent, drawn from the extremity A.

Then let AB^2 be = to the area; draw CB at right angles to it = to half the difference; from C as a centre with CB as radius, describe a circle, and draw AD through the centre.

It is evident, that the rect. DAE is = to AB^2, and ∴ to the given area.

Or thus : to rect. of sides, add sq. of half dif. and you have square of half sum ; thus, you have sum and dif. ; and thence, by addition and subtraction, you have the sides separately, and ∴ the triangle.

PROP. 30. THEOR.

If two circles touch one another internally (in C,), and if a right line (PR) *be drawn cutting both, and right lines* (PC, TC, RC, and XC,) *be drawn from the points of section to the point of contact, the angle* (PCT) *contained by one pair of them is equal to the angle* (RCX) *contained by the other pair.*

Fig. 37.

Draw a tangent LM through the point of contact ; then the angle LCP is = to the angle CRP, and the angle LCT to the angle CXT ; but CXT is = to CRX and XCR ; ∴ LCT is = to them ; take away the = parts LCP and CRP, and the remainders PCT and XCR are = ; ∴, &c. &c.

PROP. 31. PROB.

Draw two tangents to a given circle, which shall contain an angle equal to a given rectilineal angle.

Fig. 38. ANALYSIS.

Suppose it done, and that AB and AD are the required tangents, and that BAD is = to the given angle ; join the points of contact B and D with the centre I. Then, because the angles IBA and IDA are right, the angles DIB and DAB are together = to two right angles ; ∴ BID is the supplement of the given angle.

Therefore, draw a diameter ED, and from the centre I draw ID and IB making each of the angles BIE and DIC = to half the given angle, and draw the tangents BA and DA from D and B ; it is evident, that BAD is = to the given angle ; for BID is the supplement.

PROP. 32. THEOR.

If the adjacent sides of a quadrilateral figure, whose opposite angles are together equal to two right angles, be bisected by perpendiculars, and perpendiculars be drawn from their intersection to the other sides, they bisect them.

For, since the opposite angles are equal to two right angles, the figure might be inscribed in a circle ; ∴ the point of intersection of the perpendiculars is the centre ; ∴ all perpendiculars from that point, on any lines in the circle, bisect them.

PROP. 33. THEOR.

If any points (A and C) *be taken in the circumference of a given circle, and right lines be drawn from these to a point in a tangent, those drawn from them to the point of contact* (D) *contain a greater angle than those to any other point* (E).

Fig. 39.

Join AB ; then it is evident, that the angle ABC is greater than BEA ; but ABC is = ADC ; ∴, &c.

PROP. 34. PROB.

Given the segments of the base made by a right line bisecting the vertical angle, and given also the vertical angle to find the triangle.

Fig. 40.

ANALYSIS.

Suppose that EAB is the required triangle, and AC and CB the given segments ; describe about it a circle ; join EC and produce it to D.

Then, since the angles AED and BED are =, the segments AD and DB, on which they stand, are equal.

Then describe on (AB) the sum of the segts. a segment of a circle capable of containing the given angle ; complete the circle, and bisect the arc AB in D ; join DC and produce it to E ; join AE and BE ; then AEB is evidently the required triangle.

PROP. 35. THEOR.

If two tangents (AB and AD) *be drawn from the same point to a given circle and the points of contact* (BD) *be joined, the angle contained by the tangents is equal to the difference of the angles in the segments.*

Fig. 38.

For the angle SBD is = to the angles ABD and BAD ;

but the angle SBD is = to the angle BGD, and the angle ABD is = to the angle BFD ; ∴ the angle BGD is = to the angles BAD and BFD ; ∴ BAD is the difference between BGD and BFD.

PROP. 36. THEOR.

If a right line (BF) *be drawn from an angle of a regular pentagon bisecting the opposite side, it passes through the centre of the circumscribing circle.*

Fig. 35.

For the right lines BE and BD are evidently = to one another ; ∴ since BF bisects ED, it is at right angles to it ; ∴ it passes through the centre.

PROP. 37. THEOR.

If two right lines (AB and CB) *make an angle, and be cut so that the rectangle under the whole lines and segments next the angles be equal, the circle, described to pass through their extremities and one point of section* (D), *shall also pass through the other point of section.*

Fig. 41.

Draw from the point B a tangent BG ; then, if the circle does not pass through the point of section E, let it pass through F ; then the rectangles ABF and CBD, are each = to BG^2 (prop. 36, b. 3, Elr.) ; ∴ they are = to one another ; but the rect. ABE is = to the rect. CBD (by hypoth.) ; ∴ the rect. ABF is = to the rect. ABE, the greater to the less, which is absurd.

PROP. 38. THEOR.

If two chords (AB and CD) *cut one another within a given circle, the angle of their inclination is equal to half the angle at the centre, which stands on an arc equal to the sum of the arcs intercepted between them.*

And, if they cut one another outside the circle, the angle of their inclination is equal to half the angle at the centre, which stands on the difference of the arcs, which they subtend.

Fig. 42.

1. Draw BL parallel to DE, let I be the centre; then AL is = to the sum of the arcs AC and DB, and the angle of inclination AOC is = to the angle ABL (prop. 28, b. 1, Elr.); but the angle AIL, at the centre, is double of the angle ABL; ∴ the angle of inclination AOC is = to half the angle AIL.

Fig. 43.

2. If they meet outside the circle, draw DL parallel to BA; let I be the centre; join CI and LI; then the angle CDL is half of CIL; but CDL is = to COA; ∴ COA is = to half CIL.

PROP. 39. THEOR.

The vertical angle (ELF) *of any triangle* (ELF) *inscribed in a circle is greater or less than a right angle, by the angle contained by the side subtending it and the diameter drawn from the extremity of this side.*

Fig. 44.

If the given angle FLE be obtuse, join AL; then the angle ALE is = to the angle AFE, and FLA is a right angle; ∴, &c.

And, if the angle FLE be acute, draw the diameter EC; join CL.

Then the angles CLF and CEF are =; but CLE is a right angle; ∴, &c.

PROP. 40. PROB.

Three points (A, B, and C) *being given, in the circumference of a given circle, equally distant from one another, to describe two equal circles, which shall touch one another in the middle point, and pass, one through one of the extreme points, and the other through the other.*

Fig. 45.

Draw through the middle point B a tangent; join BA and BC, and make the angles BCI and BAO each respectively = to the angles ABO and CBI; the points O and I are evidently the centres of the required circles.

PROP. 41. PROB.

To describe a circle, the circumference of which shall pass through a given point (B), *and shall also touch a given right line* (DC) *in a given point* (A).

Fig. 46.

Join the given points A and B; through A draw AI at right angles to DC; bisect AB in L; through L draw LO at right angles to AB; the point O is the centre of the required circle.

Join OB; then AO is evidently = to OB; ∴ the circle, described from O as centre, with the radius OB, shall pass through the given points, and shall touch the given line in A, since the angles OAC and OAD are right.

PROP. 42. THEOR.

If from any point (A) *in the circumference of a given circle right lines* (AD, AB, and AC) *be drawn to the three angles of an inscribed equilateral triangle, the one* (AB) *to the remote angle shall be equal to the sum of the other two.*

Fig. 47.

With D as centre, and DA as radius, describe a circle cutting AB in O; join OD.

Then the angles DOA and DAO are = to one another; also the angle DAB is = to DCB (prop. 21, b. 3, Elr.); ∴ DAO is equilateral; ∴ AD is = to AO. Again, since DBC + DAC is = to two right angles, and DOB + DOA = to two right angles, DOB is = to DAC, and ADC is = to ODB, and the side DB is = to DC; ∴ AC is = to OB; ∴, &c.

PROP. 43. PROB.

To draw from the two given points (A and B) *two right lines to one point in a right line* (DE) *given in position so as to make with it two angles, whose difference shall be equal to a given angle.*

Fig. 48.

Draw AF at right angles to DE, and produce it until FG is = to AF; draw GB, and on it describe a segment of a circle capable of containing an angle = to the supplement of the given angle, and from the point C, where it intersects the given line, draw CA and CB; then the difference between the angles DCA and ECB is = to the given angle.

For, draw GCH and BC; then ACF is = to FCG, and FCG is = to ECH; ∴ ACF = ECH; ∴ ACF = ECB − HCB, and HCB is evidently = to the given angle.

PROP. 44. THEOR.

Of all the right lines, that can be drawn from two given points (AB) *to meet two by two on the convex circumference of a given circle, the least are those* (AB and PB) *that make equal angles with the tangent* (LS) *or with the radius at the point of contact.*

Fig. 49.

Let any others AI and PI be drawn; they are greater than AB and PB. For, let any point O be taken in the intercept of the tangent between AI and PI; join AO and PO.

Then AB + PB are less than AO + PO (prop. 10. on b. 1,); but AO + PO are less than AI + PI (by prop. 21. b. 1, Elr.); ∴, &c.

PROP. 45. THEOR.

Of all the triangles (ADE) *standing on the same base* (AE) *towards the same parts, and having equal vertical angles, the perimeter of that* (ACE), *which is isosceles, is the greatest.*

Fig. 50.

From C as a centre, with CA or CE as distance, describe a circle AFE; produce AC and AD to meet its circumference in F and G; join FE and GE.

Then the angles AFE and AGE are =; and also the angles ACE and ADE; but ACE is double of AFE; ∴ ADE is double of AGE, and also ADE is = to the sum of DGE and DEG; ∴ DGE is = to DEG; ∴ DG is = to DE; ∴ AG is = to AD + DE, and AF is also = to AC + CE; but AG is less than AF (prop. 3, b. 7, Elr.); ∴, &c.

PROP. 46. THEOR.

Of all the triangles (CDB and CIB) *standing on the same base* (CB), *on the same side of it, and having equal vertical angles, the greatest is that* (CDB), *which is isosceles.*

Fig. 47.

Describe a segt. of a circle on CB to pass through D; it must also pass through I; draw to the point D a tangent DH; then the angle HDB is = to the angle DCB, (prop. 32, b. 3, Elr.), and ∴ to the angle DBC; ∴ DH is parallel to CB; produce BI to meet it in H; join HC; the triangles DCB and HCB are evidently = to one another, and ICB is a part of HCB; ∴, &c.

PROP. 47. THEOR.

If the hypotenuses of two dissimilar right angled triangles (ABD and ABI) *be equal, the triangle* (AIB), *which has the smaller acute angle adjacent to its base, shall have the greatest base.*

Fig. 6.

Apply to the base of either a triangle = to the other; let it lie towards the same part with the triangle on this base; describe on this base a semicircle ADB; then because the angle BAI is less than BAD, AI is nearer to AB than AD; ∴ AI is greater than AD; ∴, &c.

PROP. 48. THEOR.

If from the centre (A) *of a circle, a right line be drawn to any point* (C) *in a chord, the square of that line, together with the rectangle under the segments* (GC and CI) *of the chord, shall be equal to the square of the radius.*

Fig. 51.

Draw any other chord BCF at right angles to AC; join AB.

Then since BC is = to CF, CB^2 will be = to the rectangle BCF; but the rect. BCF is = to the rectangle GCI; add to both AC^2; then $CA^2 + BC \times CF$ is = to $AC^2 + IC \times CG$; but $CA^2 + BC \times CF$ is = to AB^2; ∴, &c.

PROP. 49. THEOR.

The rectangle under the corresponding sides of equiangular triangles (ABC and DEF) *taken alternately are equal;* i. e. *if the angle* A = D, *and* B = E, *then* AB × EF = AC × DE.

Fig. 52.

Produce BA until AG is = to EF; then through the points B, C and G describe a circle; produce CA to meet it in H, and join HG.

Then the angles H and B are = (prop. 21, b. 3, Elr.), and also the angles GAH and CAB; ∴ the angles G and C are =; ∴ the triangle AGH is in every respect = to DEF; ∴ AH is = to DE, and AG to EF; but the rect. CAH is = to the rect. BAG; ∴ the rect. under CA and ED is = to the rect. under BA and EF; ∴, &c.

PROP. 50. THEOR.

The rectangle under two sides (AB and BC) *of any triangle is equal to the rectangle under the perpendicular* (CD) *to the base, and the diameter* (EC) *of the circumscribing circle.*

Fig. 53.

For, join BE. Then the angles A and E are =, and also the angles ADC and EBC are =, being right; ∴ the triangles EBC and ADC are similar; ∴ AC, EC, CD, and CB, being corresponding sides opposite to = angles, the rectangle under AC and CB is = to the rectangle under EC and CD by the foregoing prop.

PROP. 51. PROB.

Through a given point (P) *within a given circle to draw a right line so that the parts of it, between the given point and circumference, shall have a given difference* (F).

Fig. 54.

ANALYSIS.

Suppose EI is the line required to be drawn, join the centre A with P, and draw AO at right angles to EI; it is evident, that PO is = to half the difference of EP and PI, and ∴ to half F.

Then join AP; upon it describe a semicircle POA; inflect in it PO = to half F, and produce it to meet the circumference in E and I; EI is the line required to be drawn.

For, join OA; then, because POA is right, EI is bisected in O; ∴ OP is half the difference between EP and PI; ∴, &c.

PROP. 52. THEOR.

If from any point (B) *in the circumference of the greater of two concentric circles, right lines* (BO and BT) *be drawn to the extremities of any diameter* (OT) *of the less, the sum of the squares of those drawn lines shall be equal to double the squares of two radii of the given circles.*

Fig. 29.

For, draw BI; then, in the triangle OBT, the $\square^2$rs of OB and BT together are = to twice the $\square^2$ of BI, with twice the $\square^2$ of IO, (cor. prop. 13, b. 2, Elr.); but BI and IO are the semi-diameters of the given circles; ∴, &c.

PROP. 53. PROB.

To describe a circle whose circumference shall pass through a given point (C), *and touch a given circle in another given point* (B).

Fig. 55.

ANALYSIS.

Suppose it done, and that CBD is the circle required; join their centres; OK must pass through the point of contact B; join BC; bisect it in I; join KI; it is at right angles to BC.

If, therefore, the given points BC be joined, BC bisected in I, IK drawn at right angles to BC, the centre O of the given circle and the given point B joined, and OB be produced to meet IK, the point K, in which they meet, must be the centre of the required circle.

PROP. 54. THEOR.

If from a given point (O) *within a circle, which is not the centre, right lines* (OC, OD, and OA) *be drawn to the circumference, making equal angles with each other, those* (OD and DA), *which are next that part of the diameter* (OL) *passing through the given point, shall cut off a greater circumference than the two which are more remote.*

Fig. 56.

ANALYSIS.

Suppose that the circumference DA is greater than CD, join CD and DA; then the right line DA is greater than

CD; draw GD through the centre R; then it is evident, that OC is less than DA; make OI = to OC; join DI; then it is evident, the triangles OCD and OID are = in every respect; ∴ DI is = to DC; ∴ DA is greater than DI; if then this is proved, the proposition is true.

Produce DO and AO to S and P, and join AS, AC, and CS.

Then the angle AID is = to the angles IOD and IDO = to the angles DOC and ODC, and the angle DOC is greater than the angle OSC, and OSC is = to DAC, (prop. 21, b. 3, Elr.); ∴ DOC or DOI is greater than DAC; also the angle CDS is = to the angle CAS; ∴ CDO and COD (= IDO and IOD) are together greater than DAS; ∴ DIA is greater than DAS, and ∴ than DAI; ∴ DA is greater than DI; but DI is = to DC; ∴ DA is greater than DC.

PROP. 55. PROB.

Through a given point (A) *within a circle, which is not the centre, to draw the least possible chord.*

Fig. 57.

Find the centre K; join KA, and through A draw CP at right angles to KA; it is less than any other chord DL.

For, draw KS at right angles to DL; then, since the angle KSA is right, KA is greater than KS; ∴ CP is less than DL; in like manner it can be proved, that CO is less than any other right line drawn through A, and terminated by the circumference.

PROP. 56. PROB.

Through either of the points of intersection (A) *of two given circles, which cut each other, to draw the greatest possible line, which shall be terminated both ways by the two circumferences.*

Fig. 58.

Join the centres O and S, and draw through A the right line CD, parallel to OS; it is greater than any other right line, BF, drawn through A.

For draw OL and SI at right angles to CD; then LI is evidently half of CD; also draw OI and SQ at right angles to BF, and draw ST parallel to FB; it is also evident, that ST is half of BF; and, because ST is parallel to BF, STG is a right angle; ∴ SO is greater than ST; ∴ LI is greater than PQ, and ∴ CD is greater than BF; in like manner CD can be proved greater than any other right line drawn through A and terminated by the circumferences.

PROP. 57. THEOR.

Of all the triangles (AEC and AFC) *standing on the same base* (AC) *and having equal vertical angles, the greatest is that* (AEC) *which is isosceles.*

Fig. 59.

Let AEFC be a segment of a circle in which the = angles AEC and AFC are contained; draw EL perpendicular and FI, which is par. to AC; from the centre O draw OF, and join IA and IC.

Then it is evident, that EL bisects AC, and also passes through the centre; ∴ OF or OE is greater than OI; ∴ the triangle AEC is greater than AIC; but AIC is = to AFC; ∴ AEC is greater than AFC; ∴, &c.

PROP. 58. THEOR.

Of all the right lines (AF and EC) *that can be drawn to cut off equal areas, from a given triangle* (IDL), *that* (AF) *is the least, which makes the triangle* (IAF) *cut off isosceles.*

Fig. 60.

Let ICH be a circle passing through the points I, E, and C; bisect EC in O; join EH CH and OH.

Then the triangle ECH is isosceles, and is greater than IEC (by the foregoing prop.), and is ∴ greater than IAF; then since the triangles HEC and IAF are equiangular, and IAF less than EHC, the right line AF must consequently be greater than EC.

PROP. 59. THEOR.

If through the centre (C) *of a given circle another circle* (LCI) *be described with any radius, the intercepts* (EF and FD) *of any secant* (AD), *from the remote extre-*

mity (A) *of a right line* (AB) *joining their centres and produced to meet circumferences, are equal.*

Fig. 61.

For join CE, CF and CD; then the angles CED and CDE are = (prop. 5, b. 1. Elr.), and the angle CFE, being an angle in a semicircle, is right; ∴ CFD is right; then, in the triangles CFE and CFD, the angles CEF and CFE of one are respectively = to CDF and CFD of the other, and CE = to CD; ∴ EF is = DF (prop. 26, b. 1, Elr.); ∴, &c.

PROP. 60. THEOR.

If from a side (AC) *of a given rectangle, a part* (AI) *be cut off equal to half another side* (AE) *and from the point of section,* (I) *a right line* (IL) *be inflected equal to the cut off part* (IA), *the square of the right line* (LA), *joining their extremities, is equal to the given rectangle; and also the square of the right line* (EO) *drawn from the opposite angle at right angles to the connecting line,* (AL) *is equal to the given rectangle.*

Fig. 62.

1. For the $\square^2$ of AL is = to the sum of the $\square^2$rs of AI and IL with two rectangles under AI and IC; but the $\square^2$rs of AI and IL are together = to rect. EAI, since AI is = half AE; and two rect. AIC is = to a rect. DCI; but a rect. EAI with a rect. DCI is = to the given rect. CE; ∴, &c.

2. Also the $\square^2$ of OE is = to the rectangle CE.

For bisect AE in G; join GO, and let fall the perpendicular DF; then GO is = to GE = IL, and the angle FEO (= AOF = IAL = ILA) is = to GOE; also the angle CLA (= LAE) is = to FOF; ∴ the angle CLI is = to FOG, and LCI to OFG, and OG to IL; ∴ OF is = to LC; and since, in the triangles LCA and OFE, the angles LAC and ACL of the one are respectively = to OEF and EFO of the other, and OF = LC, OE is = to LA; but LA is = to OE; ∴ OE^2 is = to CE^2.

PROP. 61. PROB.

In a given right line (EF) *to find a point, at which right* *lines drawn from two given points* (A and H) *outside it, shall contain an angle, which shall be a maximum.*

Fig. 63.

Join AH; if it be parallel to EF, bisect it in C and draw CD at right angles to EF; join AD and HD, and the angle ADH shall be a *maximum*: for, describe a circle to pass through A, H, and D; EF is evidently a tangent to it, and then the angle ADH is evidently greater than any other angle AIH; for, join AO, and the angle AOH is = to the angle ADH; but it is greater than AIH; ∴, &c.

But, if AH is not par. to EF, produce them till they meet; find the side of a $\square^2$ = to the rect. HEA; make ED = to it, and through AHD describe a circle; ED is evidently a tangent to it; join AD and HD; the angle ADH is a *maximum*. This may be proved as the foregoing case.

PROP. 62. THEOR.

Let ADBC be a given circle, APC a given triangle inscribed in it, DP a diameter bisecting AC, and PI a perpendicular on AB; then IB shall be equal to half AB + BC.

Fig. 64.

For draw PX at right angles to BC produced, and join BP; then (prop. 26, b. 1, Elr.) BI is = to BX and PI to PX; also AP is = to PC; ∴ AI is = to CX, and ∴ IB and BX are together = to AB and BC together; but IB is = to half the former quantity, and ∴ to half the latter.

Cor. 1. Hence IB − BC = AI; and BA = BC + 2 AI; ∴ AI is = to half the difference of AB and BC.

Cor. 2. Let fall DS at right angles to AB; then AS is = to IB; for draw BH parallel to AC, and join AH, which, on account of the parallels, is = to BC; but BS, by the proposition, is = to half the difference of BA and AH or BA and BC; ∴ = AI and AS = IB.

PROP. 63. PROB.

Given the base, vertical angle and difference of the sides about it, to construct the triangle.

Fig. 65.

Describe on the given base AB a segment of a circle capable of containing the given angle; complete the circle, and draw the diameter EF at right angles to the base; join AF; on AF describe a semicircle, and in it place AO = half the given difference; produce AO to I, and join IC; IAC is the required triangle.

For, join FO; then (by the foregoing prop.) AO is = to the difference between OI and IC, and ∴ 2 AO is = to the difference between AI and IC.

PROP. 64. THEOR.

Given the sum of the sides, the difference of the segments of the base, and difference of the base angles, to find the triangle.

Fig. 73.

ANALYSIS.

Let XIO be the triangle; make XD = to the sum of the sides; join DO; then evidently the angle XOD is given = to a right angle − half the difference; and, if XY = the difference of the segments of the base, and NX = the difference of the sides, DX × XN = OX × XY; ∴ DNYO could be inscribed in a circle; ∴ the angle DNY is given; ∴ the angle XNY; and the angle NIY = the difference of the angles XIZ, and ZIO = the difference of the base angles, and is ∴ given; join IY and YD; then IY = IO = ID; ∴ the angle NDY is given, and XY is given; also DX; thence we can derive the following construction.

On XY describe a segment of a circle to contain an angle = XDY; inflect a chord = XD; join DY, and make an angle DYI = IDY; make IN = IY; let fall IZ at right angles, &c.

PROP. 65. THEOR.

Of all the right lined figures, contained under the same number of sides, and inscribed in the same circle, that is the greatest, whose sides are all equal.

Fig. 66.

For, if possible, let some polygon ABCFE, whose sides CF and FE are unequal, be the greatest.

Let CDE be an isosceles triangle in the same segment with CFE; which, being greater than CFE (prop. on b. 3.), the whole polygon ABCDE will be greater than ABCFE, which is absurd; ∴ the polygon is greatest, when all the sides are =.

PROP. 66. PROB.

To describe two circles, each having a given radius, which shall touch a given right line (DE), *both on the same side, and shall also touch one another.*

Fig. 67. ANALYSIS.

Suppose the circles CI and FI touch the given right line DE in C and F, and each other in I; join the centres L and O, and also the centres and points of contact LC and OF, and draw LH par. to DE; then it is evident, that OF is at right angles to DE; ∴, that HF is = to LC or LI, and also that LO is = to the sum of the given radii.

Then, in DE, take any point F; draw FG at right angles to DE, and = to either of the given radii; make FH = to the other; draw HL parallel to DE; and from O inflect OL = to the sum of the given radii; then the circles, described from O and L as the centres, with the given radii, shall evidently touch the given line and one another.

PROP. 67. THEOR.

If from a given point (C) *in the circumference of a circle, two right lines* (CA and CB) *be drawn to the extremities of a given chord* (AB), *the angle* (CGK), *which one of them makes with* KD *(any perpendicular to the chord,) shall be equal to the angle* (ACF), *which the other makes with a diameter* (CF) *of the circle, which passes through the given points; and also the angle* (CEK), *made by one of the drawn lines* (CA) *produced, and the perpendicular produced, is equal to the angle* (BCF) *contained by the other drawn line and diameter.*

Fig. 68.

For the angle CAK is = to the angles AEK and AKE; but the angle AKF is evidently one half AKB, and ACB is also one half AKB (prop. 20. b. 3 Elr.); ∴ CAK is

= to CEK + ECG; but CAK is = to ACK; if ∴ ECB is taken from both, BCF will remain = to KEC.

Again, the angle CGE is = to the angle GCE + GEC; but BCF is = to GEC; ∴ EGB or CGK is = to ECB + BCF or to ACF.

PROP. 68. THEOR.

A circle cannot be described about any parallelogram, which is not rectangular.

For the opposite angles of a parallelogram are = to one another; and, if a circle could be described, they would be = to two right angles; ∴ since they are =, each of them should be a right angle; but they are not so by hypothesis; ∴ a circle cannot be described.

PROP. 69. THEOR.

If two chords (AE and CB) *of a given circle, intersect one another in* (I), *the angle* (CIE) *of their inclination is equal to half the angle at the centre standing on the sum or difference of the arcs intercepted between them, according as they meet inside or outside the given circle.*

Fig. 69.

1st. Let them meet within the circle; through B draw BF parallel to AE; then the arc EF is = to AB; ∴ CEF is = to the sum of CE and AB; since BF is par. to AE, the angle CIE is = to the angle CBF; but CBF is = to half the angle at the centre standing on the arc CEF; ∴, &c.

2d. If the two chords CA and EO meet outside the circle, draw AH parallel to LE; then the arc AO is = to HE; ∴ the arc CH is the difference between AO and CE: and, since AH is parallel to LE, the angle CAH is = to CLB; ∴, &c.

PROP. 70. THEOR.

If from any point (E) *in the diameter of a circle, there be drawn two right lines* (EA and EF) *to the circumference, one to the bisection of a semicircular arc, and the other at right angles to the diameter, the squares of those two lines are together equal to two squares of the radius.*

Fig. 70.

Bisect the diameter in I; join IA and IF; then, since OA is $=$ to AL, the angles OIA and LIA are $=$, and $\therefore$ right; $\therefore$ the $\square^2$ of EA is $=$ to the $\square^2$ of IA $+$ the $\square^2$ of IE; to each add the $\square^2$ of EF; the $EA^2 + EF^2$ is $=$ to $IA^2 + IE^2 + EF^2$; but $IE^2 + EF$ is $=$ to IF^2; $\therefore EA^2 + EF^2 = IA^2 + IF^2$.

PROP. 71. THEOR.

If two circles (ACD and ECF) *touch one another, any right line* (ACF), *drawn through the point of contact, will cut off similar segments.*

Fig. 71.

Draw DCE passing through the centre of one, and the point of contact; it shall also pass through the centre of the other; join DA and EF.

Then, in the triangles DCA and FCE, the angles DCA and FCE are $=$, (prop. 15. b. 1. Elr.), and the angles DAC and EFC are $=$, being angles in semicircles; $\therefore$ the angles ADC and CEF are $=$; $\therefore$ the segments ADC and CEF are similar: in like manner the segments AIC and FLC can be proved similar.

PROP. 72. THEOR.

If two circles (ACD and ECF) *touch one another, two right lines* (ACF and DCE), *drawn through the point of contact, intercept arcs,* (DA and FE), *the chords of which are parallel.*

Fig. 71.

For, by the foregoing proposition, the arcs ADC and CEF are similar; $\therefore$ the angles ADC and CEF are $=$; but they are alternate; $\therefore$ AD is parallel to EF.

PROP. 73. THEOR.

The square of the side of an equilateral triangle (LIT), *inscribed in a circle, is equal to three times the square of the radius.*

Fig. 72.

Draw the diameter IA; it evidently bisects the angle LIT; and $\therefore$ the side LT; join LA and LF.

Then the angle LFA is = to LIT; also the angle LAF is = to LIT; ∴ LF is = to LA: but IA^2 is = to 4 FA^2 or 4 FL^2; and IA^2 is = to $IL^2 + LA^2$; and LA^2 is = to LF^2; ∴ IL^2 is = 3 FL^2.

PROP. 74. THEOR.

The right line (IA), *drawn from one angle of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the sum of the lines* (LA and TA) *drawn from the other two angles to the same point.*

Fig. 72.

Make AF = to AL; join LF.

Then, since the angles LAI and LTI are in the same segment, they are =; ∴ LAF is an angle of an equilateral triangle; also, TAI is = to ILT; ∴ LAT is = to the sum of FLA and LAF; also LFI is = to them; ∴ LFI is = to LAT; and the angles ILT and FLA being =, if FLT be taken away, ILF will be = to TLA; and since TLA is = to ILF, and LFI to LAT, and the sides LI and LT =, FI shall be = to AT; ∴ LA and AT are together = to IA.

PROP. 75. THEOR.

If perpendiculars (IL, IE, and IF) *be drawn from any point* (I) *in the circumference of a given circle to the sides of an inscribed triangle, the points* (F, E, and L), *where those perpendiculars cut the sides, shall be in directum.*

Fig. 74.

Join ID, IA, LE, and EF.

Then, since the angles IFD and IED are = to two right angles, IFDE might be inscribed in a circle; ∴ the angles IDF and IEF are =, being in the same segment; and, since the angles IEA and ILA are right, IELA might be inscribed in a circle; ∴ IEL and IAL are together = to two right angles; and, since IDBA is inscribed in a circle, the angle IDF is = to IAL; ∴ IEF is = to IAL; ∴ IEF and IEL are together = to two right angles, ∴ EF and EL are *in directum*.

PROP. 76. PROB.

Given of any triangle, the vertical angle and one of the sides containing it, to construct the triangle, when the line drawn from the vertex to the bisection of the base, makes a given angle with the base.

Fig. 75.

Let IE be = to the given side; on it describe a segment of a circle capable of containing the supplement of the given angle at the base; make IEL = to the vertical angle; bisect EI in D; draw DF parallel to EL; join IF, and produce it to meet EL in L; IEL is the required triangle.

For, IE is = to the given side and IEL to the vertical angle (by construction); also EFL is = to the given angle at the bisection of the base, since EFI is its supplement; if then the base IL be bisected at F, the thing is done; since IE is bisected in D, and DF drawn parallel to EL, IL is bisected in F (prop. 6, on b. 1); ∴, &c.

PROP. 77. THEOR.

If perpendiculars (DI and EL) *be drawn from the extremities of the diameter* (DE) *of a circle at right angles to a chord* (AB), *which is not parallel to the diameter, the greater perpendicular shall exceed the less by the intercept* (FD) *between the circumference and diameter.*

Fig. 76.

Join F and E; then DFE is a right angle; ∴ FE is par. to IL; and, since the angles DIL and ELI are right, DI is par. to LE; ∴ LE is = to IF; ∴, &c.

PROP. 78. THEOR.

The points (I and L), *where the perpendiculars, drawn from the extremities of the diameter of a circle to a chord* (AB) *produced, meet that chord, are equally distant from the centre.*

Fig. 76.

Bisect AB in T; join the centre K with T; also join F and E; since AB is bisected at T, the angles at T are right; ∴ KT is parallel to DI or LE, and FE is par. and = to IL; also FE is evidently bisected in O; ∴ IL is bisected in T; then, in the triangles ITK and LTK, the sides IT and TK are respectively = to LT and TK, and the contained angles =; ∴ KL is = to KI.

PROP. 79. THEOR.

If two circles intersect one another, and diameters be drawn from either point of intersection (I), *their extremities* (E and A) *and the other point of intersection* (F) *are in directum.*

Fig. 77. ANALYSIS.

Join EF, IF, and AF.

If EF and FA be *in directum*, the angles EFI and AFI are together = to two right angles; but they are so, since each of them is a right angle, being in a semicircle; ∴, &c.

PROP. 80. THEOR.

If a right line be drawn to touch the interior of two concentric circles, and be placed wholly within the other, it shall be bisected at the point of contact.

For, if a right line be drawn from the centre to the point of contact, it shall be at right angles to the tangent; and, since this line from the centre is at right angles to the given line, it bisects it; ∴, &c.

PROP. 81. THEOR.

Any right line (AL) *drawn from the centre of a given circle, but less than the radius, subtends the greatest possible angle at the circumference in that point* (E), *where a circle, passing through the extremities of the given line, touches the given circle.*

Fig. 78.

For, if possible, let the angle AFL be greater than it; join AI.

Then the angle AIL is greater than AFL; but AIL is = to AED; ∴ AFL is not greater than AEL; ∴, &c.

Cor. In like manner it can be proved, that any right line within a circle, not terminated on either side by the circumference, subtends the greatest possible angle at the circumference in a point thus situated.

PROP. 82. THEOR.

The shortest tangent, that can be drawn to a given circle, which shall be terminated by two given parallel right lines (EF and LK), *which are situated at different sides of the centre, is that* (IA), *which is at right angles to the given lines.*

Fig. 79.

For, if not, draw any other tangent OT less than IA; draw OH parallel to IA; since OHT is a right angle, OT is greater than OH; but IA is = to OH; ∴ OT is greater than IA; ∴, &c.

PROP. 83. PROB.

Two points (E and F) *being given, one inside and the other outside a given circle, to find a point in the circumference, to which, if two right lines be drawn from the given points, one to the concave and the other to the convex circumference, the angle contained by them shall be a minimum.*

Fig. 80.

Draw from the point E two tangents EH and EA to the given circle, (they are = to one another); join FA and FE, FH and HA; then, if the angle AEF be greater* than HEF, the angle EHF is a *minimum.*

For, since EA and EH are =, the angles EAH and EHA are =; then, in the triangles AEF and HEF, the sides AE and EF of one are respectively = to HE and EF of the other; but the angle AEF is greater than HEF; ∴ AF is greater than FH; ∴ the angle AHF is greater than HAF; ∴ FHE is less than FAE; also FHE can be proved less than any other angle FIE, one of whose sides is drawn to the concave, and the other to the convex circumference; for FIE is greater than FAE; ∴, &c.

Cor. No right line can be drawn from E to any point, such as L in the convex circumference beyond H or A; and every angle formed at any point within them can be proved greater than EHF; ∴ EAF is a *maximum.*

* The given points E and F and the centre are here supposed not to be *in directum;* for, if they were, the angles EAF and EHF would be = ; and then either of them would be less than any other angle formed on the same conditions.

PROP. 84. PROB.

To divide a given right line (AB) *so that the rectangle under the whole line and one part shall most exceed the square of that part.*

Fig. 10.

Bisect AB in C ; then the excess of the rect. BAC, above the $\square^2$ of AC, is a *maximum.*

For, take any other point D ; then the rectangle BAC is = to $AC^2 + AC \times CB$, and the rect. DAB is = to $AD^2 + AD \times DB$; but $AC \times CB$ is greater than $AD \times DB$; for, of all the equiangular parallelograms of = perimeters, that, which is equilateral, is the greatest.

PROP. 85. THEOR.

If the radius (IL) *of the lesser of two concentric circles be produced until the produced part* (LK) *be equal to it, a circle* (ETK) *being described upon this produced part as diameter, cutting the greater circle and touching the lesser ; and if a right line* (EF) *be drawn through one of the points* (E) *and point of contact* (L) *cutting the circumferences of both circles, the part of it* (LR) *within the lesser circle is equal to half the part* (EF) *within the greater.*

Fig. 82.

For, join EK, and draw IA at right angles to FE.

Then, in the triangles IAL and LEK, the angles IAL and ALI of the one are respectively = to LEK and ELK of the other ; and the side IL to LK (by hypoth.) ; ∴ AL is = to LE ; but, since IA is at right angles to RL, AL is half of RL ; for the same reason AE is half of FE ; and, since AL is = to LE, AE is = to RL ; ∴ RL is half of FE.

PROP. 86. THEOR.

If upon either side (IE) *of a quadrant, as diameter, a* *semicircle be described, any chord* (ID) *of this semicircle, drawn from the centre of the circle of this quadrant, shall be equal to the perpendicular distance* (FL) *of the point, in which the chord produced meets the arc of the quadrant, from the other radius* (IA).

Fig. 81.

For, join DE; then, in the triangles DIE and LFI, the angles FLI and IDE are =, being right; and, since AFE is a quadrant, the angle AIE is right; ∴ the angle LIF + DIE is = to the angle LIF + LFI; ∴ LFI is = to DIE, and IF is = to IE; ∴ (prop. 26, b. 1, Elr.), the triangles are = in every respect; ∴ DI is = to LF.

PROP. 87. PROB.

Given the base, altitude, and vertical angle of a triangle to construct it.

Describe on the given base a segment of a circle capable of containing the given angle; draw through its extremity a line at right angles to it, and = to the altitude; through the extremity of this perpendicular, remote from the base, draw a line parallel to it, and from a point, where this parallel line cuts the circle, draw right lines to the extremities of the given base; thus is evidently formed the triangle.

PROP. 88. THEOR.

If any two chords (IE and LF), *one in each of two given circles, be drawn through any point* (P) *in a common chord* (AD) *of both circles, the locus of their four extremities* (I, L, E and F) *shall be the circumference of a circle.*

Fig. 83.

For, describe a circle to pass through any three of them I, L, and E; it shall also pass through F; if not, let it pass through any other point K in LF.

Then, since IE and AD are chords of the same circle, intersecting one another, IP × PE is = to AP × PD; for the same reason AP × PD is = to FP × PL; and, since ILK is a circle, KP × PL is = to AP × PD; ∴ KP × PL is = to FP × PL, the greater rectangle to the less, which is absurd; ∴ the circumference does not pass through K; nor any other point in LF, but the extremity.

PROP. 89. THEOR.

The square of the right line (IL), *drawn from the obtuse angle* (EIF) *of an obtuse angled triangle to the point* (L), *in which the circumference of a semicircle, described upon the radius of the circumscribing circle, (drawn to the obtuse angle,) cuts the side* (EF) *subtending the obtuse angle, is equal to the rectangle under the segments of the base made by this line.*

Fig. 84.

Produce IL to D; join AL.

Then the angle ALI, being in a semicircle, is right; ∴ IL is = to LD; but, since EF and DI, chords of the same circle, intersect one another, the rectangle DLI is = to ELF; but the rect. DLI is = to LI^2, since DL and LI are =; ∴ the rect. ELF is = to LI^2; ∴, &c.

Cor. If a segment of a circle be given, from this proposition may be deduced a method of drawing, from any point in its circumference, a chord of the circle, that shall be bisected by the chord of the segment.

PROP. 90. THEOR.

If a circle be described on one side (CB) *of a right angled triangle* (ABC), *its circumference shall cut the hypotenuse so that the rectangle, under it and the part of it* (DB) *within the circle shall be equal to the square of side, on which the semicircle is described.*

Fig. 85.

For, since ABC is a right angle, the right line AC is a tangent to the described semicircle; ∴ the rectangle BAD is = to AC^2; but the rectangle BAD + the rectangle ABD is = to $AC^2 + CB^2$; ∴ the rectangle ABD is = to CB^2.

Cor. Thus a right line may be cut so, that the rectangle under the whole line and one segment, may be = to a given quantity.

PROP. 91. THEOR.

The rectangle under the sides (AC and CB) *of any triangle is equal to the rectangle under the perpendicular* (CD) *to the base and diameter* (CE) *of the circumscribing circle.*

Fig. 86.

Take CF = CB, and CG = CD; join FG and AE; then the angles ACE and ABE are = ; and, since the angle CDB is right, the angles DBC and DCB are together = to a right angle; ∴ EBD is = to BCD, and ∴ FCG is = to BCD; then FC and CG are respectively = to BC and CD, and the contained angles = ; ∴ the angle CGF is = to the angle CDB; ∴ CGF is a right angle, and FAE is a right angle; ∴ FAEG may be inscribed in a circle; ∴ the rectangle under AC and CF, or, under AC and CB is = to the rectangle EC and CG, or EC and CD.

Cor. The rectangle under the sides of any triangle may be proved equal to the rectangle under any secant (CE) of the circumscribed circle and the right line (CD) drawn from the vertex of the angle contained by those sides to the base, if it be drawn, so that the angle CDB shall be equal to the angle CAE.

PROP. 92. THEOR.

If the arc of any segment of a circle be bisected, the rectangle under (AI and IB) *the chords of the parts is greater than the rectangle under the chords* (AD and DB) *of any other parts, into which it may be divided.*

Fig. 97.

Because the chord is bisected in I, AI, and IB are = ; but AI and IB are together greater than AD and DB, (prop. on b. 3); ∴ AI is greater than half the sum of AD and DB, and AI^2 is greater than $\square^2$ of half the sum of AD and DB; and the $\square^2$ of half AD and DB is greater than the rectangle under AD and DB; much more then is AI^2 greater than the rectangle under AD and DB; but AI^2 is = to the rectangle under AI and IB; ∴ the rectangle under AI and IB is greater than the rectangle under AD and DB.

PROP. 93. THEOR.

If an arc (ADE) *of a given circle be bisected, and any chord* (DF) *be drawn from the point of bisection* (D) *to cut the chord* (EA) *of this arc, the rectangle under* (FD and DI) *the drawn chord and part of it intercepted between the chord of the given arc and point of bisec-*

tion, shall be equal to the square of (DE) *the chord of half the bisected arc.*

Fig. 87.

Join EF, and about the triangle IEF, describe a circle.

Then, because DE is = to DA, the angles DEA and DAE are =; but the angle DAE is also = to the angle DFE (prop. 21, b. 3, Elr.); ∴ DEA or DEI is = to EFD; ∴ DE is a tangent to the circle FEI; ∴ the rectangle under FD and DI is = to DE.

PROP. 94. THEOR.

If two circles cut one another, and two tangents (IE and IF) *be drawn from any point* (I) *in the right line* (HL) *produced, which joins their points of intersection, they shall be equal.*

Fig. 88.

For the rectangle under HI and IL is = to EI^2, and also to FI^2; ∴ EI^2 is = to FI; ∴ EI is = to FI.

Cor. 1. The right line HL, which passes through the point of section, bisects the right line DA, which touches both.

Cor. 2. Also, if the tangents IE and IF be equal, the right line HL joining the points of intersection shall pass through the common extremity (I) of the tangents.

For, join IH; it produced shall pass through D; if not, let it pass otherwise, as ILTC.

Then TI × ID is = to IE^2; also CI × IL is = to IF^2; and IF^2 is = to IE^2; ∴ CI × IL is = to TI × TL, the greater rectangle to the less, which is absurd; ∴ IL does not pass through C, nor does it pass through any other point but H.

PROP. 95. THEOR.

If a line (FL) *be drawn through the vertex of an equilateral triangle, at right angles to one of the sides,* (FA) *to meet a line* (AL) *perpendicular to another side* (AE), *it shall be equal to the radius of the circumscribing circle.*

Fig. 89.

For, draw the perpendiculars FD and ET, and produce ET to meet AL in H.

Then EI and FI are evidently equal to one another, and to the radius of the circumscribing circle; also, since LA and FD are perpendiculars, they are parallel to one another; and, since the angles LFT and FTE are right, LF is parallel to HE; ∴ HLFI is a parallelogram; ∴ HI is = to LF; and, since ED is = to DA and DI is parallel to HA; EI is = to IH, and ∴ to FL; ∴ FL is = to the radius of the circumscribing circle.

PROP. 96. THEOR.

If a circle be inscribed in any triangle (EAF), *of all the right lines, that can be drawn from the extremities of the base of the triangle to any point in the circumference, those* (EI and FI) *drawn to the point* (I), *in which a segment of a circle described on the base, touches the inscribed circle, contain the least possible angle.*

Fig. 90.

For, take any other point L, draw FL and EL; the angle EIF is less than ELF; produce EL to D; join DF; then the angles EIF and EDF are = to one another (being in the same segment); but the angle ELF is greater than EDF; in like manner EIF can be proved less than any other angle made by right lines drawn from the extremities E and F to any other point in the circumference of the inscribed circle.

PROP. 97. THEOR.

If segments of circles be described on the sides (EI and DI) *of any triangle* (EDI), *similar to this on the base* (DE), *and from the extremities of the base, tangents* (EF and DL) *be drawn to the segment on the base, and intersecting the circumferences of those on the sides, the points of intersection* (F and L) *and the vertex* (I) *of the given triangle shall be in directum.*

Fig. 91.

For, complete the segments; take any points H and C; join EH, IH, EC and DC.

Then, because FE is a tangent to the circle ICD, the angle IED is = to the angle ECD; but ECD is = to EHI, since they are in similar segments; ∴ IEH is = to EHI; but EHI and EFI are together = to two right angles (prop. b. 3, Elr.); ∴ EFI and FEH are together = to two right angles; ∴ FI is parallel to ED; for the same reason LI is parallel to ED; and, since FI and IL, drawn from the same point I, are parallel to the same right line, they must be *in directum;* ∴, &c.

PROP. 98. THEOR.

If four right lines (EF, IF, EL, and IP) *which intersect one another, and form four triangles* (ELF, ETP, IPF and ITL), *the circles circumscribing them shall all pass through one point.*

Fig. 91.

Let the circles CRTL and REPT, circumscribing two of the triangles, intersect one another in R, the circles IRF and LBT, circumscribing the other two, shall also pass through R.

For, join RI, RL, RT, RP and RE: then, since the circle CRTL passes through the points R, T, L and I, the angle RIL is = to the angle RTE; but the angle RTE is = to the angle RPE, since they are in the same segment; ∴ RPE is = to RIF; ∴ the circle passing through the points P, F and I, shall also pass through R, since the opposite angles of the figure RPFI may be proved together = to two right angles; then, because REPT is a quadrilateral figure inscribed in a circle, the angles RTI and REP are =; but RTI is = to RLI; ∴ REF is = to RLI; ∴ the angles REF and RLF are together = to two right angles; ∴ the circle passing through E, F and L shall also pass through R; ∴, &c.

PROP. 99. THEOR.

If, from any point (B) *outside a circle, there be drawn two right lines* (BI and BC), *one of which* (BC) *cuts the circle, and the other* (BI) *is at right angles to the diameter, the square of the perpendicular is equal to the rectangle under the whole secant and its external segment* (BD) *together with the rectangle contained by the segments of the diameter* (LI and IK) *made by the perpendicular.*

Fig. 41.

For, let the secant BFH be drawn through the centre T.

Then, the $\square^2$ of BT is $=$ to the $\square^2$rs of BI and IT; also BT is $=$ to $HB \times BF + TF^2$; but TF^2 is $=$ to the rectangle LI and $IK + IT^2$; $\therefore$ TB^2 is $=$ to $HB \times BF + LI \times IK + IT^2$; also BT is $=$ to $BI^2 + IT^2$; if therefore IT^2 be taken away from both, the rectangle $HB \times BF$, or $CB \times BD$ + the rectangle $LI \times IK$, is $=$ to IB^2.

PROP. 100. PROB.

Through a given point (A) *to draw a right line* (AC) *to subtend a given angle* (ABC) *so that drawing through C a line parallel to a right line given in position, the difference between AC and the said line CD terminated by AB, in D, may be given.*

Fig. 42.

Through the given angular point B draw BF parallel to the line given in position and $=$ to the given difference. Also through F draw BQ parallel to AB, meeting CB produced in P; join PA, to which apply $BR = BF$, parallel to which draw AC; it is the line required.

For, draw CDQ parallel to BF; then, because $BR = BF$, CA, being parallel to BR, must be $=$ to CQ, which is parallel to BF, and $DQ = CQ - CD$; $\therefore$ $CA - CD = BF =$ the given difference (by construction.)

PROP. 101. PROB.

From two given points (P and E), *the one P in the chord AC of a given circle ABCF, and the other E in the same chord AC produced until CE equals CP, to draw two right lines to the same point B in the periphery, and cutting it in F and D so that the rectangles under BP and PF and BD and DE may have a given difference.*

Fig. 94.

At the point C draw CG at right angles and $=$ to CE or CP; draw GE, and on it describe a semicircle, in which from G apply GH $=$ the side of a $\square^2$, that is half the given difference of the rectangles; then to the circle ABF apply $ED = EH$; produce it to B, which is the point required.

For, draw BPF and EH; then the rectangle AEC $=$ the rectangle BED $= BD \times DE + ED^2 = BD \times DE$

$+ EH^2$; add the common square GH; and $AE \times EC + GH^2 = BD \times DE + EG^2 = BD \times DE + EC^2 = BD \times DE + PE \times EC$; take away the common rectangle PEC; then $AP \times PC$ or $BP \times PF + FH^2 =$ the rectangle BDE.

PROP. 102. PROB.

From a given point P in one side CD, of a given rectangle ABCD produced, to draw a right line PS cutting the side AC in V, AB in R and DB produced in S so that the internal triangle ARV may be equal to the sum of the two external ones PCV and BRS.

Fig. 96.

From the given point P draw PH parallel to AC, and RS to AB, making the parallelogram HPDS = to twice the parallelogram ABDC; draw PS, and the thing is done.

For, the triangles HSP and DPS being each of them = the given rectangle ABDC, therefore, by taking away DBRVC common, there will remain the triangle RAV = SBR + VCP.

The construction is the same for any parallelogram.

PROP. 103. PROB.

To find a point P in an indefinite right line PQR touching a given circle in Q, at which erecting PD at right angles to PR cutting the circle in E and D, bisecting ED in I, and drawing IF parallel to PR meeting the circle in F, so that drawing DF and PF, the triangle DPF so formed may be a maximum.

Fig. 45.

Draw a tangent RCA at right angles to RP; then from R, through the centre of the given circle, draw RD; let fall DP at right angles to RQ; then P is the point required.

For, draw a tangent ADB and complete the $\square^2$ CP, which is the greatest rectangle, that can be inscribed in the right angled triangle ARB (prop. on b. 1.), and ∴ much greater than any other rectangle, having its angular point in the periphery of the given circle; because any rectangle so formed, is less than its similar one inscribed in the triangle ; whence it is evident, that the triangle PFD, being half the square PC, is the greatest possible.

PROP. 103. PROB.

To determine the locus of the point P so that, if from a given point E, EP be drawn and EL perpendicular to it, to meet the right line MR given in position, the angle ELP may be always equal to a given acute angle.

Fig. 98.

Draw EQ at right angles and ET parallel to MR, and make QEN = to the given angle (N being in MR); through N draw a right line parallel to EQ, and it shall be the required *locus.*

In NT assume any point P, and draw PL according to the proposition; on PL as diameter, describe a circle, which must pass through E and N, because the angles at those points are right, and the angle QEN = ENP; but the angles ENP and ELP are = (prop. 23. b. 3. Elr.); ∴ ENP = ELP = QEN the given angle, (by construction), and the point P is always in the same right line TN given in position.

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